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Conceptual transition systems*

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A new formalism for description of ontologies of systems and their changes – conceptual transition systems – is presented. The basic definitions of the theory of conceptual transition systems are given. These systems were demonstrated to allow to specify both typical and new kinds of ontological elements constituting ontologies. The classification of ontological elements based on such systems is described.

Keywords: transition systems, conceptual structures, ontologies, ontological elements, conceptual transition systems, conceptuals

1. Introduction

Development of formalisms, languages and tools for describing the conceptual structure of various systems is an important problem of the modern knowledge industry. Description of changes of the conceptual structure of the system when it functions is an another important problem.

Conceptual transition systems (CTSs) are a formalism of description (specification) of systems that solves these problems. This formalism is based on the following requirements:

1. It describes the conceptual structure of the specified system.
2. It describes the content of the conceptual structure of the specified system, i. e. it describes the specified system in the context of the conceptual structure.
3. It describes the change of the conceptual structure of the specified system.
4. It describes the change of the content of the conceptual structure of the specified system, i. e. it describes the change of the specified system in the context of the conceptual structure.
5. It is quite universal to specify typical ontological elements (concepts, attributes, concept instances, relations, relation instances, individuals, types, domains, and so on.).
6. It provides a quite complete classification of ontological elements, including the determination of their new kinds and subkinds.
7. It is based on the conception 'state – transition' of the usual transition systems, keeping their simplicity and universality and adding a conceptual 'filling'. This requirement is

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important since the simplicity of determination of transition systems makes them an universal formalism to describe the behavior of various systems (algorithms, programs, software models, computer systems, and so on.).

8. It supports reflection of any order, i. e. allows to specify: the system (reflection of the order 0), the specification of the system (reflection of the order 1), the specification of the specification of the system (reflection of the order 2) and so on. Specifications of the higher order (with reflection of the higher order) impose restrictions on the specifications of the lower order (with reflection of the lower order).

To our knowledge, there is no formalism that meets all the above requirements. Comparison of CTSs with the formalisms which partially meet these requirements is given in section 9.

The paper has the following structure. The preliminary concepts and notation are given in section 2. The basic definitions of the theory of CTSs are given in section 3. The classification of elements of conceptual states of CTSs such that concepts, attributes and individuals is considered in section 4. The classification of conceptuals (which can be considered as 'atoms' of conceptual states) and their associated ontological elements is presented in section 5. The ontological elements that are not directly represented in terms of elements and conceptuals of states are modelled in these terms in section 6. A generic conceptual describing sets of conceptuals matching a pattern is defined in section 7. We establish that CTSs meet the above requirements in section 8. CTSs are compared with the related formalisms in section 9.

2. Preliminaries

Let $bool = \{true, false\}$; int, nat and $nat0$ denote the sets of integers, natural numbers and natural numbers with zero, respectively; obj, fun, set, lab, arg, and val denote sets of objects, functions, sets, labels, function arguments and function values, respectively.

The names of the variables which take the values from the set with the name aw, where a is a symbol, and w is a word, are denoted by $\dot{a}w$, $\dot{a}w_1$, $\dot{a}w'$ and so forth. For example, $\dot{s}et$, $\dot{s}et_1$, $\dot{s}et'$ are the names of the variables which take the values from the set set. Depending on the context, the name of a variable is interpreted as either the variable, or the value of the variable.

Let $sup(\dot{f}un)$ and ω denote the support of $\dot{f}un$ and the indeterminate value of $\dot{f}un$, respectively.

Let $\dot{f}un(\dot{a}rg_1 \leftarrow \dot{v}al_1, \ldots, \dot{a}rg_{\dot{n}at} \leftarrow \dot{v}al_{\dot{n}at})$ denote the function $\dot{f}un'$ such that $\dot{f}un'(\dot{a}rg) = \dot{f}un(\dot{a}rg)$, if $\dot{a}rg$ is distinct from $\dot{a}rg_1$, ..., $\dot{a}rg_{\dot{n}at}$, and $\dot{f}un'(\dot{a}rg_{\dot{n}at'}) = \dot{v}al_{\dot{n}at'}$, if $1 \leq \dot{n}at' \leq \dot{n}at$.

Let $\{\dot{arg}_1 : \dot{val}_1, \ldots, \dot{arg}_{\dot{nat}} : \dot{val}_{\dot{nat}}\}$ denote the function $\dot{fun}$ such that $sup(\dot{fun}) = \{\dot{arg}_1, \ldots, \dot{arg}_{\dot{nat}}\}$, and $\dot{fun}(\dot{arg}_1) = \dot{val}_1$, ..., $\dot{fun}(\dot{arg}_{\dot{nat}}) = \dot{val}_{\dot{nat}}$. The arguments $\dot{arg}_1$, ..., $\dot{arg}_{\dot{nat}}$ are pairwise distinct.

The terms used in the paper are context-dependent. Contexts have the form $[\![\dot{obj}_1, \ldots, \dot{obj}_{\dot{nat}}]\!]$, where the embedded contexts $\dot{obj}_1$, ..., $\dot{obj}_{\dot{nat}}$ have the form: $\dot{lab}{:}\dot{obj}$, $\dot{lab}{:}$ or $\dot{obj}$.

The context in which some embedded contexts are omitted is called a partial context. All omitted embedded contexts are considered bound by the existential quantifier, unless otherwise specified.

Let $\dot{obj}[\![\dot{obj}_1, \ldots, \dot{obj}_{\dot{nat}}]\!]$ denote the object $\dot{obj}$ in the context $[\![\dot{obj}_1, \ldots, \dot{obj}_{\dot{nat}}]\!]$.

3. Basic definitions of the theory of conceptual transition systems

Let *cts* and *sys* be sets of CTSs and systems specified by these CTSs, respectively. Let $\dot{cts}$ specifies $\dot{sys}$.

3.1. The example of the specified system

Let $geoSys \in sys$ be a system which is specified by $\dot{cts}$ and is a through illustrative example of this paper.

The conceptual structure of *geoSys* includes the following entities:

- the kinds of geometric spaces (Euclidean, Riemannian, Lobachevskian and so on) specified by the labels *Euclidean*, *Riemannian*, *Lobachevskian* and so on;
- the kinds of geometric figures (triangles, rectangles, cubes and so on) specified by the concepts *triangle*, *rectangle*, *cube* and so on;
- geometric elements (certain geometric figures in a certain space) specified by individuals. Let *geoEle* be a set of geometric elements;
- the numerical characteristics of geometric figures (length, area, volume and so on) specified by the attributes *length*, *area*, *volume* and so on;
- the units (of measurement) of the numerical characteristics (inches, centimeters, metres and so on) specified by the labels *inch*, *centimeter*, *metre* and so on;
- the numeral systems for representing the values of the numerical characteristics (binary, octal, decimal and so on) specified by the natural numbers *2*, *8*, *10* and so on;
- the dimensions of geometric spaces specified by the natural numbers *1*, *2*, *3* and so on.

The change of the system *geoSys* can, for example, include various geometric transformations such that parallel a shift, rotation, homothety and so on.

3.2. Conceptual transition systems

A transition system $\dot{c}ts = (sta,\ TraRel)$ is called a conceptual transition system in $[\![ato]\!]$, if ato is a set of atoms in $[\![\dot{c}ts]\!]$, sta is a set of conceptual states in $[\![\dot{c}ts]\!]$, and $traRel \in tra \to bool$ is a transition relation in $[\![\dot{c}ts]\!]$, where $tra = sta \times sta$ is a set of transitions in $[\![\dot{c}ts]\!]$. The system $\dot{c}ts$ executes a transition $\dot{t}ra$, if $traRel(\dot{t}ra)$. The notion of conceptual state is based on notions of state, element and conceptual which are defined below.

A set ato is called a set of atoms in $[\![\dot{c}ts]\!]$, if $\omega \notin ato$, $int \subseteq ato$, and $true,\ false \in ato$. Atoms in $[\![\dot{c}ts]\!]$ are elementary 'bricks' of $\dot{c}ts$. They are used to define elements, conceptuals and states in $[\![\dot{c}ts]\!]$.

Elements in $[\![\dot{c}ts]\!]$ are basic structures of $\dot{c}ts$. In particular, they specify elements of $\dot{s}ys$. Let ele be a set of elements in $[\![\dot{c}ts]\!]$. An object $\dot{o}bj$ is called an element in $[\![\dot{c}ts]\!]$, if the following properties hold:

1. $\dot{o}bj$ is an atom in $[\![\dot{c}ts]\!]$, or
2. $\dot{o}bj$ has the form $\{\dot{l}ab_1{:}\dot{e}le_1,\ ...,\ \dot{l}ab_{\dot{n}at}{:}\dot{e}le_{\dot{n}at}\}$, where the labels $\dot{l}ab_1$, ..., $\dot{l}ab_{\dot{n}at}$ are pairwise distinct, or
3. $\dot{o}bj$ has the form $(\dot{e}le_1,\ ...,\ \dot{e}le_{\dot{n}at0})$, or
4. $\dot{o}bj$ has the form $\{\dot{e}le_1,\ ...,\ \dot{e}le_{\dot{n}at0}\}$.

Elements of the forms 2, 3, and 4 are called labelled, ordered and unordered (element) structures, respectively. Let $labStr$, $ordStr$, $unoStr$ and $eleStr = labStr \cup ordStr \cup unoStr$ be sets of labelled structures, ordered structures, unordered structures and element structures, respectively. The elements $()$ and $\{\}$ are called empty structures (the empty ordered structure and the empty unordered structure, respectively).

Let $1 \leq \dot{n}at' \leq \dot{n}at$. Let $\dot{e}leStr(\dot{n}at')$ and $\dot{e}leStr(\%lab_{\dot{n}at'})$ denote $\dot{e}le_{\dot{n}at'}$, if $\dot{e}leStr$ has the form $(\dot{e}le_1,\ ...,\ \dot{e}le_{\dot{n}at})$ and $\{lab_1{:}\dot{e}le_1,\ ...,\ lab_{\dot{n}at}{:}\dot{e}le_{\dot{n}at}\}$, respectively.

The function $len \in ele \to nat0$ is called a length in $[\![ele{:}]\!]$, if $len(\dot{a}to) = 0$, and $len(\dot{e}leStr)$ is the number of elements in $\dot{e}leStr$.

The equality operation $=$ on elements is defined as follows: $\dot{e}le = \dot{e}le'$ if and only if $len(\dot{e}le) = len(\dot{e}le') = \dot{n}at$, and

- $\dot{e}le$ and $\dot{e}le'$ are equal atoms, or

- $\dot{ele} = (\dot{ele}_1, \ldots, \dot{ele}_{\dot{n}at0})$, $\dot{ele}' = (\dot{ele}'_1, \ldots, \dot{ele}'_{\dot{n}at0})$, $\dot{ele}_1 = \dot{ele}'_1$, ..., $\dot{ele}_{\dot{n}at0} = \dot{ele}'_{\dot{n}at0}$, or
- $\dot{ele} = \{\dot{ele}_1, \ldots, \dot{ele}_{\dot{n}at0}\}$, $\dot{ele}' = \{\dot{ele}'_1, \ldots, \dot{ele}'_{\dot{n}at0}\}$, $\dot{ele}_1 = \dot{ele}'_1$, ..., $\dot{ele}_{\dot{n}at0} = \dot{ele}'_{\dot{n}at0}$, or
- $\dot{ele} = \{\dot{lab}_1{:}\dot{ele}_1, \ldots, \dot{lab}_{\dot{n}at}{:}\dot{ele}_{\dot{n}at}\}$, $\dot{ele}' = \{\dot{lab}'_1{:}\dot{ele}'_1, \ldots, \dot{lab}'_{\dot{n}at}{:}\dot{ele}'_{\dot{n}at}\}$, $\dot{lab}_1 = \dot{lab}'_1$, ..., $\dot{lab}_{\dot{n}at} = \dot{lab}'_{\dot{n}at}$, $\dot{ele}_1 = \dot{ele}'_1$, ..., $\dot{ele}_{\dot{n}at} = \dot{ele}'_{\dot{n}at}$.

Conceptuals in $[\![\dot{cts}]\!]$ are the special kind of elements which specify ontological elements of $\dot{sys}$. An element $\dot{labStr}$ is called a conceptual in $[\![\dot{cts}]\!]$, if all its labels are integers. Let con be a set of conceptuals.

Example. Let $\dot{con} = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$. Then the following properties hold:

- $\dot{con}$ is a conceptual in $\dot{cts}$;
- $\dot{con}$ specifies the area (the label -1) of the triangle (the label 1) $\dot{g}eoEle$ (the label 0) in three-dimensional (the label 3) Euclidean (the label 2) space, measured in inches (the label -2) in the decimal system (the label -3).

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A function $\dot{fun} \in con \to ele$ is called a conceptual state in $[\![\dot{cts}]\!]$. A state in $[\![\dot{cts}]\!]$ is called conceptual because it specifies the conceptual structure of the system $\dot{sys}$, associating conceptuals with their values.

A function $sem \in con \times sta \to ele$ is called a semantics in $[\![con{:}]\!]$, if $sem(\dot{con}, \dot{sta}) = \dot{sta}(\dot{con})$. The element $sem(\dot{con}, \dot{sta})$ is called the value in $[\![\dot{con}, \dot{sta}]\!]$.

Example. Let $\dot{con} = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{sta}(\dot{con}) = 3$. Then the following properties hold:

- $sem(\dot{con}, \dot{sta}) = 3$;
- 3 is the value in $[\![\dot{con}, \dot{sta}]\!]$;
- the area of the triangle $\dot{g}eoEle$ in two-dimensional Euclidean space is equal to 3 inches in the decimal system in $[\![\dot{sta}]\!]$.

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3.3. Structure of conceptuals

An element $\dot{ele}$ is called an element in $[\![\dot{con}, \dot{int}]\!]$, if $\dot{ele}= \dot{con}(\dot{int})$. A number $\dot{int}$ is called an element order in $[\![\dot{con}, \dot{ele}]\!]$, if $\dot{ele}= \dot{con}(\dot{int})$. Let $eleOrd$ be a set of element orders.

Example. Let $\dot{con} = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$. Then the following properties hold:

- *10*, *inch*, *area*, $\dot{g}eoEle$, *triangle*, *Euclidean*, *2* are elements in $[\![\dot{c}on]\!]$ in $[\![-3]\!]$, $[\![-2]\!]$, $[\![-1]\!]$, $[\![0]\!]$, $[\![1]\!]$, $[\![2]\!]$, $[\![3]\!]$, respectively;
- -3, -2, -1, 0, 1, 2, 3 are element orders in $[\![\dot{c}on]\!]$ in $[\![10]\!]$, $[\![inch]\!]$, $[\![area]\!]$, $[\![\dot{g}eoEle]\!]$, $[\![triangle]\!]$, $[\![Euclidean]\!]$, $[\![3]\!]$, respectively.

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Proposition 1. The value ω is not an element in $[\![\dot{c}on]\!]$.

Proof. This follows from the fact that ω is not an element in $[\![\dot{c}ts]\!]$. □

Proposition 2. The number of elements in $[\![\dot{c}on]\!]$ is finite.

Proof. This follows from the fact that $sup(\dot{c}on)$ is finite, and ω is not an element in $[\![\dot{c}on]\!]$. □

Proposition 3. If $\dot{e}le$ is an element in $[\![\dot{c}on]\!]$, then the number of element orders in $[\![\dot{c}on, \dot{e}le]\!]$ is finite.

Proof. This follows from the fact that $sup(\dot{c}on)$ is finite, and ω is not an element in $[\![\dot{c}on]\!]$. □

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}10)$. Then the following properties hold:

- -3 and 3 are element orders in $[\![\dot{c}on, 10]\!]$;
- if $\dot{e}le$ is an element in $[\![\dot{c}on]\!]$ which is distinct from *10*, then there is the unique element order in $[\![\dot{c}on, \dot{e}le]\!]$.

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Proposition 4. The number of element orders in $[\![\dot{c}on]\!]$ is finite.

Proof. This follows from the fact that $sup(\dot{c}on)$ is finite. □

An order $\dot{e}leOrd[\![\dot{c}on, \dot{e}le]\!]$ is called a minimal element order in $[\![\dot{c}on, \dot{e}le]\!]$, if $\dot{i}nt$ is not an element order in $[\![\dot{c}on, \dot{e}le]\!]$ for each $\dot{i}nt$ such that $\dot{i}nt < \dot{e}leOrd$. An order $\dot{e}leOrd[\![\dot{c}on]\!]$ is called a minimal element order in $[\![\dot{c}on]\!]$, if $\dot{i}nt$ is not an element order in $[\![\dot{c}on]\!]$ for each $\dot{i}nt$ such that $\dot{i}nt < \dot{e}leOrd$. An element $\dot{e}le$ is called a minimal element in $[\![\dot{c}on]\!]$, if there exists $\dot{e}leOrd[\![\dot{c}on, \dot{e}le]\!]$ such that $\dot{e}leOrd$ is a minimal element order in $[\![\dot{c}on]\!]$.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}10)$. Then the following properties hold:

- -3, -2, -1, 0, 1, 2, 3 are element orders in $[\![\dot{c}on]\!]$ in $[\![10]\!]$, $[\![inch]\!]$, $[\![area]\!]$, $[\![\dot{g}eoEle]\!]$, $[\![triangle]\!]$, $[\![Euclidean]\!]$, $[\![10]\!]$, respectively;
- -3 is a minimal element order in $[\![\dot{c}on]\!]$;

- *10* is a minimal element in $[\![\dot{con}]\!]$.

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An order $\dot{eleOrd}[\![\dot{con},\ \dot{ele}]\!]$ is called a maximal element order in $[\![\dot{con},\ \dot{ele}]\!]$, if $\dot{int}$ is not an element order in $[\![\dot{con},\ \dot{ele}]\!]$ for each $\dot{int}$ such that $\dot{eleOrd} < \dot{int}$. An order $\dot{eleOrd}[\![\dot{con}]\!]$ is called a maximal element order in $[\![\dot{con}]\!]$, if $\dot{int}$ is not an element order in $[\![\dot{con}]\!]$ for each $\dot{int}$ such that $\dot{eleOrd} < \dot{int}$. An element $\dot{ele}$ is called a maximal element in $[\![\dot{con}]\!]$, if there exists $\dot{eleOrd}[\![\dot{con},\ \dot{ele}]\!]$ such that $\dot{eleOrd}$ is a maximal element order in $[\![\dot{con}]\!]$.

Example. Let $\dot{con}$ = *(−3:10, −2:inch, −1:area, 0:ġeoEle, 1:triangle, 2:Euclidean, 3:10)*. Then the following properties hold:

- *−3, −2, −1, 0, 1, 2, 3* are element orders in $[\![\dot{con}]\!]$ in $[\![10]\!]$, $[\![inch]\!]$, $[\![area]\!]$, $[\![\dot{g}eoEle]\!]$, $[\![triangle]\!]$, $[\![Euclidean]\!]$, *10*, respectively;
- *3* is a maximal element order in $[\![\dot{con}]\!]$;
- *10* is a maximal element in $[\![\dot{con}]\!]$;
- *10* is both minimal and maximal element in $[\![\dot{con}]\!]$.

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An element $\dot{ele}$ is called a null element in $[\![\dot{con}]\!]$, if $\dot{ele}$ is an element in $[\![\dot{con},\ 0]\!]$.

Example. Let $\dot{con}$ = *(−3:10, −2:inch, −1:area, 0:ġeoEle, 1:triangle, 2:Euclidean, 3:2)*. Then *ġeoEle* is a null element in $[\![\dot{con}]\!]$. •

3.4. Conceptuals and elements of states

A conceptual $\dot{con}$ is called a conceptual in $[\![\dot{sta}]\!]$, if $sem(\dot{con},\ \dot{sta}) \neq \omega$.

Example. Let $\dot{con}$ = *(−3:10, −2:inch, −1:area, 0:ġeoEle, 1:triangle, 2:Euclidean, 3:2)*, and $\dot{sta}(\dot{con}) = 3$. Then $\dot{con}$ is a conceptual in $[\![\dot{sta}]\!]$. •

An element $\dot{ele}$ is called an element in $[\![\dot{sta},\ \dot{int},\ \dot{con}[\![\dot{sta}]\!]]\!]$, if $\dot{ele}$ is an element in $[\![\dot{con},\ \dot{int}]\!]$. The number $\dot{int}$ is called an order in $[\![\dot{ele},\ \dot{sta},\ \dot{con}]\!]$. The conceptual $\dot{con}$ is called a concretization conceptual in $[\![\dot{ele},\ \dot{sta},\ \dot{int}]\!]$.

Example. Let $\dot{con}$ = *(−3:10, −2:inch, −1:area, 0:ġeoEle, 1:triangle, 2:Euclidean, 3:2)*, and $\dot{sta}(\dot{con}) = 3$. Then the following properties hold:

- *10, inch, area, ġeoEle, trianle, Euclidean, 3* are elements in $[\![\dot{sta}]\!]$ in $[\![-3]\!]$, $[\![-2]\!]$, $[\![-1]\!]$, $[\![0]\!]$, $[\![1]\!]$, $[\![2]\!]$, $[\![3]\!]$ in $[\![\dot{con}]\!]$, respectively;
- *−3, −2, −1, 0, 1, 2, 3* are orders in $[\![10]\!]$, $[\![inch]\!]$, $[\![area]\!]$, $[\![\dot{g}eoEle]\!]$, $[\![triangle]\!]$, $[\![Euclidean]\!]$, $[\![2]\!]$ in $[\![\dot{sta}]\!]$ in $[\![\dot{con}]\!]$, respectively;

- $\dot{con}$ is a concretization conceptual in $[\![10]\!]$, $[\![inch]\!]$, $[\![area]\!]$, $[\![\dot{g}eoEle]\!]$, $[\![triangle]\!]$, $[\![Euclidean]\!]$, $[\![2]\!]$ in $[\![\dot{sta}]\!]$ in $[\![-3]\!]$, $[\![-2]\!]$, $[\![-1]\!]$, $[\![0]\!]$, $[\![1]\!]$, $[\![2]\!]$, $[\![3]\!]$, respectively.

$\bullet$

Proposition 5. For all $\dot{ele}$ and $\dot{int}$ there exist $\dot{sta}$ and $\dot{con}[\![\dot{sta}]\!]$ such that $\dot{ele}$ is an element in $[\![\dot{sta}, \dot{int}, \dot{con}]\!]$.

Proof. We define $\dot{sta}$ and $\dot{con}$ as follows: $\dot{con}(\dot{int}) = \dot{ele}$, and $\dot{sta}(\dot{con}) \neq \omega$. Then $\dot{ele}$ is an element in $[\![\dot{sta}, \dot{int}, \dot{con}]\!]$. $\square$

4. Classification of elements of states

Elements in $[\![\dot{sta}]\!]$ are subclassified into individuals, concepts and attributes.

Individuals in $[\![\dot{sta}]\!]$ specify elements of $\dot{sys}$. An element $\dot{ele}[\![\dot{sta}]\!]$ is called an individual in $[\![\dot{sta}, \dot{con}[\![\dot{sta}]\!]]\!]$, if $\dot{ele}$ is an element in $[\![\dot{sta}, 0, \dot{con}]\!]$.

Example. Let $\dot{con} = (-3{:}10, -2{:}inch, -1{:}area, 0{:}\dot{g}eoEle, 1{:}triangle, 2{:}Euclidean, 3{:}2)$, and $\dot{sta} = (\dot{con}{:}3)$. Then $\dot{g}eoEle$ is an individual in $[\![\dot{sta}]\!]$ in $[\![\dot{con}]\!]$. $\bullet$

Concepts in $[\![\dot{sta}]\!]$ generalizes the usual concepts of the ontology of *sys* which are interpreted as sets of elements of *sys*. An element $\dot{ele}[\![\dot{sta}]\!]$ is called a concept in $[\![\dot{sta}, \dot{nat}, \dot{con}[\![\dot{sta}]\!]]\!]$, if $\dot{ele}$ is an element in $[\![\dot{sta}, \dot{nat}, \dot{con}]\!]$. Let *conc* be a set of concepts.

Example. Let $\dot{con} = (-3{:}10, -2{:}inch, -1{:}area, 0{:}\dot{g}eoEle, 1{:}triangle, 2{:}Euclidean, 3{:}2)$, and $\dot{sta} = (\dot{con}{:}3)$. Then *triangle*, *Euclidean*, *3* are concepts in $[\![\dot{sta}]\!]$ in $[\![1]\!]$, $[\![2]\!]$, $[\![3]\!]$ in $[\![\dot{con}]\!]$, respectively. $\bullet$

Attributes in $[\![\dot{sta}]\!]$ generalizes the usual attributes of the ontology of *sys* which are interpreted as characteristics of elements of *sys*. An element $\dot{ele}[\![\dot{sta}]\!]$ is called an attribute in $[\![\dot{sta}, \dot{nat}, \dot{con}[\![\dot{sta}]\!]]\!]$, if $\dot{ele}$ is an element in $[\![\dot{sta}, -\dot{nat}, \dot{con}]\!]$. A number $\dot{nat}$ is called an order in $[\![atr{:}\dot{ele}, \dot{sta}, \dot{con}]\!]$. The label *atr* is used to distinguish orders of concepts from orders of attributes, since the element $\dot{ele}$ can be both a concept and an attribute in $[\![\dot{sta}, \dot{nat}, \dot{con}]\!]$. The conceptual $\dot{con}$ is called a concretization conceptual in $[\![atr{:}\dot{ele}, \dot{sta}, \dot{nat}]\!]$. Let *atr* be a set of attributes.

Example. Let $\dot{con} = (-3{:}10, -2{:}inch, -1{:}area, 0{:}\dot{g}eoEle, 1{:}triangle, 2{:}Euclidean, 3{:}2)$, and $\dot{sta} = (\dot{con}{:}3)$. Then the following properties hold:

- *area*, *inch*, *10* are attributes in $[\![\dot{sta}]\!]$ in $[\![1]\!]$, $[\![2]\!]$, $[\![3]\!]$ in $[\![\dot{con}]\!]$, respectively;
- *1*, *2*, *3* are orders in $[\![atr{:}area]\!]$, $[\![atr{:}inch]\!]$, $[\![atr{:}10]\!]$ in $[\![\dot{sta}]\!]$ in $[\![\dot{con}]\!]$, respectively;
- $\dot{con}$ is a concretization conceptual in $[\![atr{:}area]\!]$, $[\![atr{:}inch]\!]$, $[\![atr{:}10]\!]$ in $[\![\dot{sta}]\!]$ in $[\![1]\!]$, $[\![2]\!]$,

$[\![3]\!]$, respectively.

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Concepts and attributes are considered in detail below.

4.1. Concepts

The usual concepts of the ontology of $\dot{s}ys$ which are interpreted as sets of elements of $\dot{s}ys$ are specified by the special kind of concepts in $[\![\dot{s}ta]\!]$ – direct concepts in $[\![\dot{s}ta]\!]$. An element $\dot{e}le[\![\dot{s}ta]\!]$ is called a direct concept in $[\![\dot{s}ta,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{e}le$ is a concept in $[\![\dot{s}ta,\ 1,\ \dot{c}on]\!]$. Let *dirConc* be a set of direct concepts.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then *triangle* is a direct concept in $[\![\dot{s}ta]\!]$ in $[\![\dot{c}on]\!]$. It specifies the element $\dot{g}eoEle$ as a triangle in $[\![\dot{s}ta]\!]$. •

An element $\dot{e}le$ is called an element in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{c}onc$ is a concept in $[\![\dot{s}ta,\ \dot{n}at_1,\ \dot{c}on]\!]$, $\dot{e}le$ is an element in $[\![\dot{c}on,\ \dot{n}at_2]\!]$, and $\dot{n}at_2 < \dot{n}at_1$. Thus, elements of the concept $\dot{c}onc$ can be concepts of orders which are less than the order of $\dot{c}onc$, individuals and attributes of any orders.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then the following properties hold:

1. *10*, *inch*, *area*, $\dot{g}eoEle$ are elements in $[\![conc{:}triangle]\!]$ in $[\![\dot{s}ta]\!]$ in $[\![concOrd{:}1]\!]$ in $[\![eleOrd{:}\text{-}3]\!]$, $[\![eleOrd{:}\text{-}2]\!]$, $[\![eleOrd{:}-1]\!]$, $[\![eleOrd{:}0]\!]$ in $[\![\dot{c}on]\!]$, respectively. In particular, this means that the triangle $\dot{g}eoEle$ has the area which is measured in inches represented in the decimal system in $[\![\dot{s}ta]\!]$;
2. *10*, *inch*, *area*, $\dot{g}eoEle$, *triangle* are elements in $[\![conc{:}Eucludian]\!]$ in $[\![\dot{s}ta]\!]$ in $[\![concOrd{:}2]\!]$ in $[\![eleOrd{:}\text{-}3]\!]$, $[\![eleOrd{:}\text{-}2]\!]$, $[\![eleOrd{:}-1]\!]$, $[\![eleOrd{:}0]\!]$, $[\![eleOrd{:}1]\!]$ in $[\![\dot{c}on]\!]$, respectively. In particular, this means that the triangle $\dot{g}eoEle$ belongs to Euclidean space in $[\![\dot{s}ta]\!]$;
3. *10*, *inch*, *area*, $\dot{g}eoEle$, *triangle*, *Eucludian* are elements in $[\![conc{:}2]\!]$, in $[\![\dot{s}ta]\!]$ in $[\![concOrd{:}3]\!]$ in $[\![eleOrd{:}-3]\!]$, $[\![eleOrd{:}-2]\!]$, $[\![eleOrd{:}-1]\!]$, $[\![eleOrd{:}0]\!]$, $[\![eleOrd{:}1]\!]$, $[\![eleOrd{:}2]\!]$ in $[\![\dot{c}on]\!]$, respectively. In particular, this means that the triangle $\dot{g}eoEle$ belongs to two-dimensional space in $[\![\dot{s}ta]\!]$.

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Proposition 6. If $\dot{c}onc$ is a concept in $[\![\dot{s}ta]\!]$, and $\dot{e}le$ is an element in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}1]\!]$, then $\dot{e}le$ is either an individual in $[\![\dot{s}ta]\!]$, or $\dot{e}le$ is an attribute in $[\![\dot{s}ta]\!]$.

Proof. This follows from the definition of direct concepts. □

A set $\dot{s}et$ is called the content in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{s}et$ is a set of elements in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$. The content of a concept describes its semantics.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then the following properties hold:

- $\{2,\ inch,\ area,\ \dot{g}eoEle\}$ is the content in $[\![conc{:}triangle]\!]$ in $[\![\dot{s}ta]\!]$;
- $\{2,\ inch,\ area,\ \dot{g}eoEle,\ triangle\}$ is the content in $[\![conc{:}Eucludian]\!]$ in $[\![\dot{s}ta]\!]$;
- $\{2,\ inch,\ area,\ \dot{g}eoEle,\ triangle,\ Eucludian\}$ is the content in $[\![conc{:}2]\!]$ in $[\![\dot{s}ta]\!]$.

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Example. Let $\dot{c}on_1 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}3)$, $\dot{c}on_2 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_2,\ 1{:}triangle,\ 2{:}Riemannian,\ 3{:}3)$, $\dot{c}on_3 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on_1{:}3,\ \dot{c}on_2{:}4,\ \dot{c}on_3{:}2)$. Then the following properties hold:

- $\{\dot{g}eoEle_1,\ \dot{g}eoEle_2\}$ is the content in $[\![conc{:}triangle,\ \dot{s}ta,\ concOrd{:}1,\ eleOrd{:}0]\!]$. This means that the individuals $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ are triangles in $[\![\dot{s}ta]\!]$;
- $\{triangle\}$ is the content in $[\![conc{:}Eucludian]\!]$, $[\![conc{:}Riemannian]\!]$ in $[\![\dot{s}ta,\ concOrd{:}2,\ eleOrd{:}1]\!]$, respectively. This means that Euclidean and Riemannian spaces can include triangles in $[\![\dot{s}ta]\!]$;
- $\{Eucludian,\ Riemannian\}$ is the content in $[\![conc{:}3,\ \dot{s}ta,\ concOrd{:}3,\ eleOrd{:}2]\!]$. This means that three-dimensional space can be either Euclidean or Riemannian in $[\![\dot{s}ta]\!]$;
- $\{\dot{g}eoEle_1\}$ is the content in $[\![conc{:}2,\ \dot{s}ta,\ concOrd{:}3,\ eleOrd{:}0]\!]$. This means that two-dimensional space includes the individual $\dot{g}eoEle_1$ in $[\![\dot{s}ta]\!]$.

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An element $\dot{e}le$ is called an element in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!],\ med{:}\dot{n}at0]\!]$, if $\dot{e}le$ is an element in $[\![\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$, and $\dot{n}at0$ is the number of element orders $\dot{n}at$ in $[\![\dot{c}on]\!]$ such that $\dot{n}at_2 < \dot{n}at < \dot{n}at_1$. The integer $\dot{n}at0$ is called a mediatorial decree in $[\![\dot{e}le,\ \dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$. It specifies how many mediators separate $\dot{e}le$ from $\dot{c}onc$ in $\dot{c}on$. The element $\dot{e}le'$ is called a mediator in $[\![\dot{e}le,\ \dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{e}le'$ is an element in $[\![\dot{c}on,\ \dot{n}at]\!]$, and $\dot{n}at_2 < \dot{n}at < \dot{n}at_1$.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then $\dot{g}eoEle$ is an element in the following contexts:

- $[\![conc{:}triangle,\ \dot{s}ta,\ concOrd{:}1,\ eleOrd{:}0,\ \dot{c}on]\!]$ with the mediatorial decree *0* and without mediators;
- $[\![conc{:}Euclidean,\ \dot{s}ta,\ concOrd{:}2,\ eleOrd{:}0,\ \dot{c}on]\!]$ with the mediatorial decree *1* and the mediator *triangle*;
- $[\![conc{:}2,\ \dot{s}ta,\ concOrd{:}3,\ eleOrd{:}0,\ \dot{c}on]\!]$ with the mediatorial decree *2* and the mediators *triangle* and *Euclidean*.

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An element $\dot{e}le$ is called a direct element in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{e}le$ is an element in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on,\ med{:}0]\!]$.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then the following properties hold:

- $\dot{g}eoEle$ is a direct element in $[\![conc{:}triangle,\ \dot{s}ta,\ concOrd{:}1,\ eleOrd{:}0]\!]$;
- *triangle* is a direct element in $[\![conc{:}Eucludian,\ \dot{s}ta,\ concOrd{:}2,\ eleOrd{:}1]\!]$;
- *Eucludian* is a direct element in $[\![conc{:}2,\ \dot{s}ta,\ concOrd{:}3,\ eleOrd{:}2]\!]$.

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Example. Let $\dot{c}on_1 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}3)$, $\dot{c}on_2 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_2,\ 1{:}triangle,\ 2{:}Riemannian,\ 3{:}3)$, $\dot{c}on_3 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on_1{:}3,\ \dot{c}on_2{:}4,\ \dot{c}on_3{:}2)$. Then the following properties hold:

1. $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ are direct elements in $[\![conc{:}triangle,\ \dot{s}ta]\!]$;
2. *triangle* is a direct element in $[\![conc{:}Eucludian]\!]$ and $[\![conc{:}Riemannian]\!]$ in $[\![\dot{s}ta]\!]$;
3. *Eucludian* and *Riemannian* are direct elements in $[\![conc{:}3,\ \dot{s}ta]\!]$;
4. $\dot{g}eoEle_1$ is a direct element in $[\![conc{:}2,\ \dot{s}ta]\!]$.

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A set $\dot{s}et$ is called the direct content in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{s}et$ is a set of direct elements in $[\![conc{:}\dot{c}onc,\ \dot{s}ta,\ concOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$.

Example. Let $\dot{c}on_1 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}3)$, $\dot{c}on_2 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_2,\ 1{:}triangle,\ 2{:}Riemannian,\ 3{:}3)$, $\dot{c}on_3 = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on_1{:}3,\ \dot{c}on_2{:}4,\ \dot{c}on_3{:}2)$. Then the following properties hold:

- $\{\dot{g}eoEle_1,\ \dot{g}eoEle_2\}$ is the direct content in $[\![conc{:}triangle,\ \dot{s}ta]\!]$;

- *{triangle}* is the direct content in $[\![$*conc:Eucludian*$]\!]$, $[\![$*conc:Riemannian*$]\!]$ in $[\![\dot{s}ta]\!]$, respectively;
- *{Eucludian, Riemannian}* is the direct content in $[\![$*conc:3,* $\dot{s}ta]\!]$;
- *{*$\dot{g}eoEle_1$*}* is the direct content in $[\![$*conc:2,* $\dot{s}ta]\!]$.

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A set $\dot{s}et$ is called the content in $[\![$*conc:*$\dot{c}onc$*,* $\dot{s}ta$*, concOrd:*$\dot{n}at_1$*, eleOrd:*$\dot{n}at_2$*,* $\dot{c}on[\![\dot{s}ta]\!]$*, med:*$\dot{n}at_3]\!]$, if $\dot{s}et$ is a set of elements in $[\![$*conc:*$\dot{c}onc$*,* $\dot{s}ta$*, concOrd:*$\dot{n}at_1$*, eleOrd:*$\dot{n}at_2$*,* $\dot{c}on$*, med:*$\dot{n}at_3]\!]$.

Example. Let $\dot{c}on_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_2$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_2$*, 1:triangle, 2:Riemannian, 3:2)*, $\dot{c}on_3$ = *(−3:10, −2:inch, −1:perimeter, 0:*$\dot{g}eoEle_3$*, 2:Euclidean, 3:2)*, and $\dot{s}ta$ = *(*$\dot{c}on_1$*:3,* $\dot{c}on_2$*:4,* $\dot{c}on_3$*:2)*. Then the following properties hold:

- *{*$\dot{g}eoEle_1$*,* $\dot{g}eoEle_2$*}* is the content in $[\![$*conc:2,* $\dot{s}ta$*, concOrd:3, eleOrd:0, med:2*$]\!]$;
- *{*$\dot{g}eoEle_3$*}* is the content in $[\![$*conc:2,* $\dot{s}ta$*, concOrd:3, eleOrd:0, med:1*$]\!]$;
- *{area}* is the content in $[\![$*conc:2,* $\dot{s}ta$*, concOrd:3, eleOrd:−1, med:3*$]\!]$;
- *{perimeter}* is the content in $[\![$*conc:2,* $\dot{s}ta$*, concOrd:3, eleOrd:−1, med:2*$]\!]$.

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4.2. Classification and interpretation of concepts

Concepts are classified according to their orders.

A concept $\dot{c}onc$ in $[\![\dot{s}ta$*, concOrd:1*$]\!]$ specifies a usual concept of the ontology of $\dot{s}ys$. Elements in $[\![$*conc:*$\dot{c}onc$*,* $\dot{s}ta$*, concOrd:1)*$]\!]$ are attributes and individuals in $[\![\dot{s}ta]\!]$.

Example. Let $\dot{c}on_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_2$ = *(−3:2, −2:cm, −1:perimeter, 0:*$\dot{g}eoEle_2$*, 1:triangle, 2:Euclidean, 3:2)*, and $\dot{s}ta$ = *(*$\dot{c}on_1$*:3,* $\dot{c}on_2$*:4)*. Then the following properties hold:

1. The direct concept *triangle* specifies triangles in $[\![\dot{s}ta]\!]$.
2. The individuals $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ are elements of the order *0* of the direct concept *triangle* in $[\![\dot{s}ta]\!]$. This means that $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ are triangles in $[\![\dot{s}ta]\!]$.
3. The attributes *area* and *perimeter* are elements of the order *−1* of the direct concept *triangle* in $[\![\dot{s}ta]\!]$. This means that triangles can have area and perimeter in $[\![\dot{s}ta]\!]$.
4. The attributes *inch* and *cm* are elements of the order *−2* of the direct concept *triangle* in $[\![\dot{s}ta]\!]$. This means that numerical characteristics of triangles can be measured in inches

and centimetres in $[\![\dot{sta}]\!]$.

5. The attributes *10* and *2* are elements of the order -3 of the direct concept *triangle* in $[\![\dot{sta}]\!]$. This means that the values of numerical characteristics of triangles can be represented in decimal and binary systems in $[\![\dot{sta}]\!]$.

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A concept $\dot{conc}$ in $[\![\dot{sta}, concOrd{:}2]\!]$ specifies a concept space of the ontology of $\dot{sys}$. Elements in $[\![conc{:}\dot{conc}, \dot{sta}, concOrd{:}2]\!]$ are attributes, individuals and direct concepts in $[\![\dot{sta}]\!]$.

Example. Let $\dot{con}_1 = (-3{:}10, -2{:}inch, -1{:}area, 0{:}\dot{geoEle}_1, 1{:}triangle, 2{:}Euclidean, 3{:}2)$, $\dot{con}_2 = (-3{:}2, -2{:}cm, -1{:}perimeter, 0{:}\dot{geoEle}_2, 1{:}square, 2{:}Euclidean, 3{:}2)$, and $\dot{sta} = (\dot{con}_1{:}3, \dot{con}_2{:}4)$. Then the following properties hold:

1. The concept space *Euclidean* specifies Euclidean space in $[\![\dot{sta}]\!]$.
2. The direct concepts *triangle* and *square* are elements of the order *1* of the concept space *Euclidean* in $[\![\dot{sta}]\!]$. This means that triangles and squares can belong to Euclidean space in $[\![\dot{sta}]\!]$.
3. The individuals $\dot{geoEle}_1$ and $\dot{geoEle}_2$ are elements of the order *0* of the concept space *Euclidean* in $[\![\dot{sta}]\!]$. This means that the geometric elements $\dot{geoEle}_1$ and $\dot{geoEle}_2$ belong to Euclidean space in $[\![\dot{sta}]\!]$.
4. The attributes *area* and *perimeter* are elements of the order -1 of the concept space *Euclidean* in $[\![\dot{sta}]\!]$. This means that geometric elements from Euclidean space can have area and perimeter in $[\![\dot{sta}]\!]$.
5. The attributes *inch* and *cm* are elements of the order -2 of the concept space *Euclidean* in $[\![\dot{sta}]\!]$. This means that numerical characteristics of geometric elements from Euclidean space can be measured in inches and centimetres in $[\![\dot{sta}]\!]$.
6. The attributes *10* and *2* are elements of the order -3 of the concept space *Euclidean* in $[\![\dot{sta}]\!]$. This means that values of numerical characteristics of geometric elements from Euclidean space can be represented in decimal and binary systems in $[\![\dot{sta}]\!]$.

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A concept $\dot{conc}$ in $[\![\dot{sta}, concOrd{:}3]\!]$ specifies a space of concept spaces of the ontology of $\dot{sys}$. Elements in $conc{:}\dot{conc}, \dot{sta}, concOrd{:}3$ are attributes, individuals, direct concepts and concept spaces in $[\![\dot{sta}]\!]$.

Example. Let $\dot{con}_1 = (-3{:}10, -2{:}inch, -1{:}area, 0{:}\dot{geoEle}_1, 1{:}triangle, 2{:}Euclidean, 3{:}2)$, $\dot{con}_2 = (-3{:}2, -2{:}cm, -1{:}perimeter, 0{:}\dot{geoEle}_2, 1{:}square, 2{:}Riemannian, 3{:}2)$, and $\dot{sta} =$

$(\dot{con}_1{:}3,\ \dot{con}_2{:}4)$. Then the following properties hold:

1. The concept space space *2* specifies two-dimensional space in $[\![\dot{sta}]\!]$.
2. The concept spaces *Euclidean* and *Riemannian* are elements of the order *2* of the concept space space *2* in $[\![\dot{sta}]\!]$. This means that Euclidean and Riemannian spaces can be two-dimensional in $[\![\dot{sta}]\!]$.
3. The direct concepts *triangle* and *square* are elements of the order *1* of the concept space space *2* in $[\![\dot{sta}]\!]$. This means that triangles and squares can belong to two-dimensional space in $[\![\dot{sta}]\!]$.
4. The individuals $\dot{geoEle}_1$ and $\dot{geoEle}_2$ are elements of the order *0* of the concept space space *2* in $[\![\dot{sta}]\!]$. This means that geometric elements $\dot{geoEle}_1$ and $\dot{geoEle}_2$ belong to two-dimensional space in $[\![\dot{sta}]\!]$.
5. The attributes *area* and *perimeter* are elements of the order -1 of the concept space space *2* in $[\![\dot{sta}]\!]$. This means that geometric elements from two-dimensional space can have area and perimeter in $[\![\dot{sta}]\!]$.
6. The attributes *inch* and *cm* are elements of the order -2 of the concept space space *2* in $[\![\dot{sta}]\!]$. This means that numerical characteristics of geometric elements from two-dimensional space can be measured in inches and centimetres in $[\![\dot{sta}]\!]$.
7. The attributes *10* and *2* are elements of the order -3 of the concept space space *2* in $[\![\dot{sta}]\!]$. This means that values of numerical characteristics of geometric elements from two-dimensional space can be represented in decimal and binary systems in $[\![\dot{sta}]\!]$.

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A concept $\dot{conc}$ in $[\![\dot{sta},\ concOrd{:}\dot{int}]\!]$, where $\dot{int} > 3$, is classified and interpreted in the similar way (by the introduction of the space of concept space spaces and so on.).

4.3. The attributes

Attributes use the same terminology as concepts.

The usual attributes of the ontology of $\dot{sys}$ which are interpreted as characteristics of elements of $\dot{sys}$ are specified by the special kind of attributes in $[\![\dot{sta}]\!]$ – direct attributes in $[\![\dot{sta}]\!]$.

An element $\dot{ele}[\![\dot{sta}]\!]$ is called a direct attribute in $[\![\dot{sta},\ \dot{con}[\![\dot{sta}]\!]]\!]$, if $\dot{ele}$ is an attribute in $[\![\dot{sta},\ concOrd{:}1,\ \dot{con}]\!]$.

Example. Let $\dot{con} = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{geoEle},\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{sta} = (\dot{con}{:}3)$. Then the following properties hold:

- *area* is a direct attribute in $[\![\dot{s}ta,\ \dot{c}on]\!]$. It specifies the individual $\dot{g}eoEle$ as the element which has an area in $[\![\dot{s}ta]\!]$;
- *area* is a direct attribute in $[\![\dot{s}ta]\!]$. It specifies the set of elements which have an area in $[\![\dot{s}ta]\!]$.

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An element $\dot{e}le$ is called an element in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{a}tt$ is an attribute in $[\![\dot{s}ta,\ \dot{n}at_1,\ \dot{c}on]\!]$, $\dot{e}le$ is an element in $[\![\dot{c}on,\ \dot{n}at_2]\!]$, and $-\dot{n}at_1 < \dot{n}at_2$. Thus, elements of the attribute $\dot{a}tt$ can be attributes of orders which is less than the order of $\dot{a}tt$, individuals and concepts of all orders.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then the following properties hold:

1. $\dot{g}eoEle$, *triangle*, *Euclidean* and *2* are elements in $[\![att{:}area,\ \dot{s}ta,\ attOrd{:}1]\!]$ in $[\![eleOrd{:}0]\!]$, $[\![eleOrd{:}1]\!]$, $[\![eleOrd{:}2]\!]$, $[\![eleOrd{:}3]\!]$ in $[\![\dot{c}on]\!]$, respectively. In particular, this is means that the triangle $\dot{g}eoEle$ from two-dimensional Euclidean space has an area in $[\![\dot{s}ta]\!]$.
2. *area*, $\dot{g}eoEle$, *triangle*, *Euclidean* and *2* are elements in $[\![att{:}inch,\ \dot{s}ta,\ attOrd{:}2]\!]$ in $[\![eleOrd{:}-1]\!]$, $[\![eleOrd{:}0]\!]$, $[\![eleOrd{:}1]\!]$, $[\![eleOrd{:}2]\!]$, $[\![eleOrd{:}3]\!]$ in $[\![\dot{c}on]\!]$, respectively. In particular, this means that the area of the triangle $\dot{g}eoEle$ from two-dimensional Euclidean space is measured in inches in $[\![\dot{s}ta]\!]$.
3. *inch*, *area*, $\dot{g}eoEle$, *triangle*, *Euclidean* and *2* are elements in $[\![att{:}10,\ \dot{s}ta,\ attOrd{:}3]\!]$ in $[\![eleOrd{:}-2]\!]$, $[\![eleOrd{:}-1]\!]$, $[\![eleOrd{:}0]\!]$, $[\![eleOrd{:}1]\!]$, $[\![eleOrd{:}2]\!]$, $[\![eleOrd{:}3]\!]$ in $[\![\dot{c}on]\!]$, respectively. This means that the area of the triangle $\dot{g}eoEle$ from two-dimensional Euclidean space measured in inches is represented in the decimal system in $[\![\dot{s}ta]\!]$.

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Proposition 7. If $\dot{a}tt$ is an attribute in $[\![\dot{s}ta]\!]$, and $\dot{e}le$ is an element in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}1]\!]$, then $\dot{e}le$ is either an individual or a concept in $[\![\dot{s}ta]\!]$.

Proof. This follows from the definition of direct attributes. □

A set $\dot{s}et$ is called the content in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{s}et$ is a set of elements in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$. The content of an attribute describes its semantics.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then the following properties hold:

- $\{\dot{g}eoEle,\ triangle,\ Euclidean,\ 2\}$ is the content in $[\![att{:}area,\ \dot{s}ta]\!]$;

- $\{area,\ \dot{g}eoEle,\ triangle,\ Euclidean,\ 2\}$ is the content in $[\![att{:}inch,\ \dot{s}ta]\!]$;
- $\{inch,\ area,\ \dot{g}eoEle,\ triangle,\ Euclidean,\ 2\}$ is the content in $[\![att{:}10,\ \dot{s}ta]\!]$.

•

An element $\dot{e}le$ is called an element in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!],\ med{:}\dot{n}at0]\!]$, if $\dot{e}le$ is an element in $[\![\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$, and $\dot{n}at0$ is the number of element orders $\dot{n}at$ in $[\![\dot{c}on]\!]$ such that $\dot{n}at_2 < \dot{n}at < \dot{n}at_1$. A number $\dot{n}at0$ is called a mediatorial degree in $[\![\dot{e}le,\ att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$. It specifies how many mediators separate $\dot{e}le$ from $\dot{a}tt$ in $\dot{c}on$. An element $\dot{e}le'$ is called a mediator in $[\![\dot{e}le,\ att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{e}le'$ is an element in $[\![\dot{c}on,\ \dot{n}at]\!]$, and $\dot{n}at_2 < \dot{n}at < \dot{n}at_1$.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then $\dot{g}eoEle$ is an element in the following contexts:

- $[\![att{:}area,\ \dot{s}ta,\ attOrd{:}1,\ eleOrd{:}0,\ \dot{c}on]\!]$ with the mediatorial degree 0 and without mediators;
- $[\![att{:}inch,\ \dot{s}ta,\ attOrd{:}2,\ eleOrd{:}0,\ \dot{c}on]\!]$ with the mediatorial degree 1 and the mediator *area*;
- $[\![att{:}10,\ \dot{s}ta,\ attOrd{:}3,\ eleOrd{:}0,\ \dot{c}on]\!]$ with the mediatorial degree 2 and the mediators *area* and *inch*.

•

An element $\dot{e}le$ is called a direct element in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{e}le$ is an element in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on,\ med{:}0]\!]$.

Example. Let $\dot{c}on = (-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, and $\dot{s}ta = (\dot{c}on{:}3)$. Then the following properties hold:

1. $\dot{g}eoEle$ is a direct element in $[\![att{:}area,\ \dot{s}ta,\ attOrd{:}1,\ eleOrd{:}0]\!]$. This means that the individual $\dot{g}eoEle$ has an area in $[\![\dot{s}ta]\!]$.
2. *area* is a direct element in $[\![att{:}inch,\ \dot{s}ta,\ attOrd{:}2,\ eleOrd{:}1]\!]$. This means that an area can be measured in inches in $[\![\dot{s}ta]\!]$.
3. *inch* is a direct element in $[\![att{:}10,\ \dot{s}ta,\ attOrd{:}3,\ eleOrd{:}2]\!]$. This means that values of numerical characteristics of geometric elements measured in inches can be represented in the decimal system in $[\![\dot{s}ta]\!]$.

•

A set $\dot{s}et$ is called the direct content in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on[\![\dot{s}ta]\!]]\!]$, if $\dot{s}et$ is a set of direct elements in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}\dot{n}at_1,\ eleOrd{:}\dot{n}at_2,\ \dot{c}on]\!]$.

Example. Let $\dot{c}on_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_2$ = *(−3:10, −2:cm, −1:area, 0:*$\dot{g}eoEle_2$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_3$ = *(−3:2, 0:*$\dot{g}eoEle_1$*, 3:2)*, and $\dot{s}ta$ = *(*$\dot{c}on_1$*:3,* $\dot{c}on_2$*:4,* $\dot{c}on_3$*:2).* Then the following properties hold:

- *{*$\dot{g}eoEle_1$*,* $\dot{g}eoEle_2$*}* is the direct content in ⟦*att:area,* $\dot{s}ta$⟧;
- *{area}* is the direct content in ⟦*att:inch*⟧ and ⟦*att:cm*⟧ in ⟦$\dot{s}ta$⟧;
- *{inch, cm}* is the direct content in ⟦*att:10,* $\dot{s}ta$⟧;
- *{*$\dot{g}eoEle_1$*}* is the direct content in ⟦*att:2,* $\dot{s}ta$⟧.

•

A set $\dot{s}et$ is called the content in ⟦*att:*$\dot{a}tt$*,* $\dot{s}ta$*, concOrd:*$\dot{n}at_1$*, eleOrd:*$\dot{n}at_2$*,* $\dot{c}on$⟦$\dot{s}ta$⟧*, med:*$\dot{n}at0$⟧, if $\dot{s}et$ is a set of elements in ⟦*att:*$\dot{a}tt$*,* $\dot{s}ta$*, attOrd:*$\dot{n}at_1$*, eleOrd:*$\dot{n}at_2$*,* $\dot{c}on$*, med:*$\dot{n}at0$⟧.

Example. Let $\dot{c}on_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_2$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_2$*, 1:triangle, 2:Riemannian, 3:2)*, $\dot{c}on_3$ = *(−3:10, −2:inch, 0:*$\dot{g}eoEle_3$*, 1:square, 2:Euclidean, 3:2)*, and $\dot{s}ta$ = *(*$\dot{c}on_1$*:3,* $\dot{c}on_2$*:4,* $\dot{c}on_3$*:2).* Then the following properties hold:

- *{*$\dot{g}eoEle_1$*,* $\dot{g}eoEle_2$*}* is the content in ⟦*att:10,* $\dot{s}ta$*, attOrd:3, eleOrd:0, med:2*⟧;
- *{*$\dot{g}eoEle_3$*}* is the content in ⟦*att:10,* $\dot{s}ta$*, attOrd:3, eleOrd:0, med:1*⟧;
- *{triangle}* is the content in ⟦*att:10,* $\dot{s}ta$*, concOrd:3, eleOrd:1, med:3*⟧;
- *{square}* is the content in ⟦*att:10,* $\dot{s}ta$*, attOrd:3, eleOrd:1, med:2*⟧.

•

4.4. Classification and interpretation of attributes

Attributes are classified according to their orders.

An attribute $\dot{a}tt$ in ⟦$\dot{s}ta$*, attOrd:1*⟧ specifies a usual attribute of the ontology of $\dot{s}ys$. Elements in ⟦*att:*$\dot{a}tt$*,* $\dot{s}ta$*, attOrd:1*⟧ are individuals and concepts in ⟦$\dot{s}ta$⟧.

Example. Let $\dot{c}on_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_2$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_2$*, 1:square, 2:Riemannian, 3:3)*, and $\dot{s}ta$ = *(*$\dot{c}on_1$*:3,* $\dot{c}on_2$*:4).* Then the following properties hold:

1. The direct attribute *area* specifies an area of geometric elements in ⟦$\dot{s}ta$⟧.
2. The individuals $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ are elements of the order *0* of the direct attribute *area* in ⟦$\dot{s}ta$⟧. This means that $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ has an area in ⟦$\dot{s}ta$⟧.
3. The concepts *triangle* and *square* are elements of the order *1* of the direct attribute *area* in ⟦$\dot{s}ta$⟧. This means that triangles and squares can have an area in ⟦$\dot{s}ta$⟧.

4. The concept spaces *Euclidean* and *Riemannian* are elements of the order *2* of the direct attribute *area* in $[\![\dot{s}ta]\!]$. This means that numerical characteristics of geometric elements from Euclidean and Riemannian spaces can have an area in $[\![\dot{s}ta]\!]$.
5. The concept space spaces *2* and *3* are elements of the order *3* of the direct attribute *area* in $[\![\dot{s}ta]\!]$. This means that values of numerical characteristics of geometric elements from two-dimensional and three-dimensional spaces can have an area in $[\![\dot{s}ta]\!]$.

•

An attribute $\dot{a}tt$ in $[\![\dot{s}ta,\ attOrd{:}2]\!]$ specifies an attribute space of the ontology of $\dot{s}ys$. Elements in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}2]\!]$ are direct attributes, individuals and concepts in $[\![\dot{s}ta]\!]$.

Example. Let $\dot{c}on_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_2$ = *(−3:10, −2:inch, −1:perimeter, 0:*$\dot{g}eoEle_2$*, 1:square, 2:Riemannian, 3:3)*, and $\dot{s}ta$ = *(*$\dot{c}on_1$*:3,* $\dot{c}on_2$*:4)*. Then the following properties hold:

1. The attribute space *inch* specifies numerical characteristics of geometric elements measured in inches in $[\![\dot{s}ta]\!]$.
2. The direct attributes *area* and *perimeter* are elements of the order −1*1* of the attribute space *inch* in $[\![\dot{s}ta]\!]$. This means that areas and perimeters of geometric elements can be measured in inches in $[\![\dot{s}ta]\!]$.
3. The individuals $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ are elements of the order *0* of the attribute space *inch* in $[\![\dot{s}ta]\!]$. This means that geometric elements $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ can have numerical characteristics measured in inches in $[\![\dot{s}ta]\!]$.
4. The concepts *triangle* and *square* are elements of the order *1* of the attribute space *inch* in $[\![\dot{s}ta]\!]$. This means that numerical characteristics of triangles and squares can be measured in inches in $[\![\dot{s}ta]\!]$.
5. The concept spaces *Euclidean* and *Riemannian* are elements of the order *2* of the attribute space *inch* in $[\![\dot{s}ta]\!]$. This means that numerical characteristics of geometric elements from Euclidean and Riemannian spaces can be measured in inches in $[\![\dot{s}ta]\!]$.
6. The concept space spaces *2* and *3* are elements of the order *3* of the attribute space *inch* in $[\![\dot{s}ta]\!]$. This means that numerical characteristics of geometric elements from two-dimensional and three-dimensional spaces can be measured in inches in $[\![\dot{s}ta]\!]$.

•

An attribute $\dot{a}tt$ in $[\![\dot{s}ta,\ attOrd{:}3]\!]$ specifies a space of attribute spaces of the ontology of $\dot{s}ys$. Elements in $[\![att{:}\dot{a}tt,\ \dot{s}ta,\ attOrd{:}1]\!]$ are attribute spaces, direct attributes, individuals and concepts in $[\![\dot{s}ta]\!]$.

Example. Let $\dot{c}on_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{c}on_2$ = *(−3:10, −2:cm, −1:perimeter, 0:*$\dot{g}eoEle_2$*, 1:square, 2:Riemannian, 3:3)*, and $\dot{s}ta$ = *(*$\dot{c}on_1$*:3,* $\dot{c}on_2$*:4)*. Then the following properties hold:

1. The attribute space space *10* specifies geometric elements, values of numerical characteristics of which are represented in the decimal system.
2. The attribute spaces *inch* and *cm* are elements of the order −*2* of the attribute space space *10* in $[\![\dot{s}ta]\!]$. This means that values of numerical characteristics of geometric figures measured in inches can be represented in the decimal system in $[\![\dot{s}ta]\!]$.
3. The direct attributes *area* and *perimeter* are elements of the order −1*1* of the attribute space space *10* in $[\![\dot{s}ta]\!]$. This means that values of areas and perimeters of geometric elements can be represented in the decimal system in $[\![\dot{s}ta]\!]$.
4. The individuals $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ are elements of the order *0* of the attribute space space *10* in $[\![\dot{s}ta]\!]$. This means that values of numerical characteristics of geometric elements $\dot{g}eoEle_1$ and $\dot{g}eoEle_2$ can be represented in the decimal system in $[\![\dot{s}ta]\!]$.
5. The concepts *triangle* and *square* are elements of the order *1* of the attribute space space *10* in $[\![\dot{s}ta]\!]$. This means that values of numerical characteristics of triangles and squares can be represented in the decimal system in $[\![\dot{s}ta]\!]$.
6. The concept spaces *Euclidean* and *Riemannian* are elements of the order *2* of the attribute space space *10* in $[\![\dot{s}ta]\!]$. This means that numerical characteristics of geometric elements from Euclidean and Riemannian spaces can be represented in the decimal system in $[\![\dot{s}ta]\!]$.
7. The concept space spaces *10* and *2* are elements of the order *3* of the attribute space space *10* in $[\![\dot{s}ta]\!]$. This means that the values of numerical characteristics of geometric elements from two-dimensional and three-dimensional spaces can be represented in the decimal system in $[\![\dot{s}ta]\!]$.

•

An attribute $\dot{a}tt$ in $[\![\dot{s}ta,\ attOrd{:}\dot{n}at]\!]$, where $\dot{n}at > 3$, is classified and interpreted in the similar way (by the introduction of spaces of attribute space space and so on.).

4.5. Notes about elements of states

Concepts and attributes are opposite (symmetric in some respects) entities. Concepts generalize (combine into groups) elements of $\dot{c}ts$. In contrast, attributes concretize (divide into sub-elements) elements of $\dot{c}ts$.

In addition to specification of elements of $\dot{s}ys$, an individual $\dot{e}le[\![\dot{s}ta]\!]$ can be interpreted in two ways:

- in the attribute context, $\dot{e}le$ is interpreted as an attribute in $[\![\dot{s}ta,\ attOrd{:}0]\!]$. In this case, it specifies a global attribute of $\dot{s}ys$;
- in the concept context, $\dot{e}le$ is interpreted as a concept in $[\![\dot{s}ta,\ concOrd{:}0]\!]$. In this case, it specifies a concept which has the single instance $\dot{e}le$.

5. Classification of conceptuals

5.1. General principles and definitions

The two-level scheme of classification of conceptuals is used. The upper (first) level is defined by the maximal order of attributes of a conceptual. This level is described by the notion of concretization order of a conceptual. The lower (second) level is defined by the set of all element orders of a conceptual. This level is described by the notion of integral order of a conceptual.

5.1.1. Concretization orders of conceptuals

The number *0* is called an order in $[\![\dot{c}on]\!]$, if the minimal order in $\dot{c}on$ is greater than or equal to *0*. A number $\dot{n}at$ is called an order in $[\![\dot{c}on]\!]$, if $-\dot{n}at$ is a minimal order in $\dot{c}on$.

Example. Let $\dot{c}on_1$ = $(-3{:}10,\ -2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, $\dot{c}on_2 = (-2{:}inch,\ -1{:}area,\ 0{:}\dot{g}eoEle_1,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, $\dot{c}on_3 = (-1{:}area,\ 0{:}\dot{g}eoEle_1,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, $\dot{c}on_4 = (0{:}\dot{g}eoEle_1,\ 1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, $\dot{c}on_5 = (1{:}triangle,\ 2{:}Euclidean,\ 3{:}2)$, $\dot{c}on_6 = (2{:}Euclidean,\ 3{:}2)$, and $\dot{c}on_7 = (3{:}2)$. Then the following properties hold:

1. The conceptuals $\dot{c}on_1$, $\dot{c}on_2$, $\dot{c}on_3$ have the orders *3*, *2*, *1*, respectively.
2. The conceptuals $\dot{c}on_4$, $\dot{c}on_5$, $\dot{c}on_6$, $\dot{c}on_7$ have the order *0*.

•

Conceptuals of the order $\dot{n}at$ concretizes conceptuals of the orders which are less than $\dot{n}at$. They define the special kinds of such conceptuals and are used to classify them. Concretization is performed by attributes of the order $\dot{n}at$ and their values. Therefore, the order of a conceptual is also called the concretization order of the conceptual.

5.1.2. Integral orders of conceptuals

A set $\dot{set}$ is called an integral order in $[\![\dot{con}]\!]$, if $\dot{set}$ is a set of all element orders in $[\![\dot{con}]\!]$. Let $intOrd$ be a set of integral orders.

Proposition 8. A conceptual $\dot{con}$ has the single integral order.

Proof. This follows from the definition of the integral order of a conceptual. □

Example. Let $\dot{con}_1$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*, $\dot{con}_1$ = *(−3:10, −1:area, 1:triangle, 3:2)*, $\dot{con}_1$ = *(−2:inch, −1:area, 2:Euclidean, 3:2)*. Then $\dot{intOrd}[\![\dot{con}_1]\!]$ = *{−3, −2, −1, 0, 1, 2, 3}*, $\dot{intOrd}[\![\dot{con}_2]\!]$ = *{−3, −1, 1, 3}*, and $\dot{intOrd}[\![\dot{con}_3]\!]$ = *{−2, −1, 2, 3}*. •

A set $\dot{set}$ is called a refined integral order in $[\![\dot{con}]\!]$, if $\dot{set}$ is a result of replacement of zero or more element orders $\dot{eleOrd}[\![\dot{con}]\!]$ in the set $\dot{intOrd}[\![\dot{con}]\!]$ by objects $\dot{eleOrd}$*:*$\dot{con}(\dot{eleOrd})$. A refined integral order in $\dot{con}$ refines an integral order in $\dot{con}$, providing information on some elements of $\dot{con}$ with their orders. Let $\dot{con}:\dot{intOrd}$ denote a conceptual $\dot{con}$ which has the refined integral order $\dot{intOrd}$.

Example. Let $\dot{con}$ = *(−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2)*. Then *{−3, −2, −1, 0, 1, 2, 3}*, *{−3, −2:inch, −1, 0, 1:triangle, 2, 3}* and *{−3:10, −2:inch, −1:area, 0:*$\dot{g}eoEle_1$*, 1:triangle, 2:Euclidean, 3:2}* are refined integral orders in $[\![\dot{con}]\!]$. •

Proposition 9. A conceptual $\dot{con}$ has a finite set of refined integral orders.

Proof. This follows from the definition of the refined integral order and the finite number of element orders of conceptuals. □

Proposition 10. The integral order in $[\![\dot{con}]\!]$ is a refined integral order in $[\![\dot{con}]\!]$.

Proof. This follows from the definition of the refined integral order of a conceptual. □

Conceptuals of the same concretization order are classified according to their integral orders. Each integral order defines a separate kind of conceptuals.

Conceptuals allow to classify ontological elements in detail. Each kind of conceptuals specifies a separate kind of ontological elements.

5.2. Correlation between ontological elements and conceptuals of the order *0*

In this section conceptuals of the order *0* is classified according to their integral orders and kinds of conceptuals of this classification are correlated with the corresponding kinds of ontological elements.

A conceptual $\dot{con}$*:{0}* specifies the individual $\dot{con}$*(0)*.

Example. The conceptual *(0:ġeoEle)* specifies the geometric element *ġeoEle*. •

A conceptual *ċon:{0, 1}* specifies the individual *ċon(0)* from the concept *ċon(1)*.

Example. The conceptual *(0:ġeoEle, 1:triangle)* specifies the triangle *ġeoEle*. •

A conceptual *ċon:{1}* specifies the concept *ċon(1)*.

Example. A conceptual *(1:triangle)* specifies triangles. •

A conceptual *ċon:{1, 2}* specifies the concept *ċon(1)* from the concept space *ċon(2)*.

Example. The conceptual *(1:triangle, 2:Euclidean)* specifies triangles from Euclidean space. •

A conceptual *ċon:{2}* specifies the concept space *ċon(2)*.

Example. The conceptual *(2:Euclidean)* specifies Euclidean space. •

A conceptual *ċon:{0, 2}* specifies the individual *ċon(0)* from the concept space *ċon(2)*.

Example. The conceptual *(0:ġeoEle, 2:Euclidean)* specifies the geometric element *ġeoEle* from Euclidean space. •

A conceptual *ċon:{0, 1, 2}* specifies the individual *ċon(0)* from the concept *ċon(1)* from the concept space *ċon(2)*.

Example. The conceptual *(0:ġeoEle, 1:triangle, 2:Euclidean)* specifies the triangle *ġeoEle* from Euclidean space. •

Correlation between other kinds of conceptuals of the order *0* and the corresponding kinds of ontological elements is performed in a similar way. For example, a conceptual *ċon:{0, 1, 2, 3}* specifies the individual *ċon(0)* from the concept *ċon(1)* from the concept space *ċon(2)* from the concept space space *ċon(3)*.

Example. The conceptual *(0:ġeoEle, 1:triangle, 2:Euclidean, 3:2)* specifies the triangle *ġeoEle* from two-dimensional Euclidean space. •

5.3. Correlation between ontological elements and conceptuals of the order *1*

In this section conceptuals of the order *1* is classified according to their integral orders and kinds of conceptuals of this classification are correlated with the corresponding kinds of ontological elements.

A conceptual *ċon:{-1}* specifies the attribute *ċon(-1)*.

Example. The conceptual *(−1:area)* specifies an area of geometric elements. •

A conceptual *ċon:{-1, 0}* specifies the attribute *ċon(-1)* of the individual *ċon(0)*.

Example. The conceptual *(−1:area, 0:ġeoEle)* specifies the area of the geometric element *ġeoEle*. •

A conceptual *ċon:{-1, 0, 1}* specifies attribute *ċon(-1)* of the individual *ċon(0)* from the concept *ċon(1)*.

Example. The conceptual *(−1:area, 0:ġeoEle, 1:triangle)* specifies an area of the triangle *ġeoEle*. •

A conceptual *ċon:{-1, 1}* specifies attribute *ċon(-1)* of the concept *ċon(1)*.

Example. The conceptual *(−1:area, 1:triangle)* specifies areas of triangles. •

A conceptual *ċon:{-1, 0, 1, 2}* specifies attribute *ċon(-1)* of the individual *ċon(0)* from the concept *ċon(1)* from the concept space *ċon(2)*.

Example. The conceptual *(−1:area, 0:ġeoEle, 1:triangle, 2:Euclidean)* specifies the area of the triangle *ġeoEle* from Euclidean space. •

A conceptual *ċon:{-1, 1, 2}* specifies attribute *ċon(-1)* of the concept *ċon(1)* from the concept space *ċon(2)*.

Example. The conceptual *(−1:area, 1:triangle, 2:Euclidean)* specifies areas of triangles from Euclidean space. •

A conceptual *ċon:{-1, 0, 2}* specifies attribute *ċon(-1)* of the individual *ċon(0)* from the concept space *ċon(2)*.

Example. The conceptual *(−1:area, 0:ġeoEle, 2:Euclidean)* specifies the area of the geometric element *ġeoEle* from Euclidean space. •

A conceptual *ċon:{-1, 2}* specifies attribute *ċon(-1)* of the concept space *ċon(2)*.

Example. The conceptual *(−1:area, 2:Euclidean)* specifies areas of geometric elements from Euclidean space. •

Correlation between other kinds of conceptuals of the order *1* and the corresponding kinds of ontological elements is performed in a similar way.

5.4. Correlation between ontological elements and conceptuals of the order *2*

In this section conceptuals of the order *2* is classified according to their integral orders and kinds of conceptuals of this classification are correlated with the corresponding kinds of ontological elements.

A conceptual *ċon:{-2, -1}* specifies the attribute *ċon(-1)* in the attribute space *ċon(-2)*.

Example. The conceptual *(−2:inch, −1:area)* specifies areas in inches. •

A conceptual *ċon:{-2, -1, 0}* specifies the attribute *ċon(-1)* of the individual *ċon(0)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, −1:area, 0:ġeoEle)* specifies the area of the geometric element *ġeoEle* in inches. •

A conceptual *ċon:{-2, -1, 0, 1}* specifies the attribute *ċon(-1)* of the individual *ċon(0)* from the concept *ċon(1)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, −1:area, 0:ġeoEle, 1:triangle)* specifies the area of the triangle *ġeoEle* in inches. •

A conceptual *ċon:{-2, -1, 1}* specifies the attribute *ċon(-1)* of the concept *ċon(1)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, −1:area, 1:triangle)* specifies areas of triangles in inches. •

A conceptual *ċon:{-2, -1, 0, 1, 2}* specifies the attribute *ċon(-1)* of the individual *ċon(0)* from the concept *ċon(1)* from the concept space *ċon(2)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, −1:area, 0:ġeoEle, 1:triangle, 2:Euclidean)* specifies the area of the triangle *ġeoEle* from Euclidean space in inches. •

A conceptual *ċon:{-2, -1, 1, 2}* specifies the attribute *ċon(-1)* of the concept *ċon(1)* from the concept space *ċon(2)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, −1:area, 1:triangle, 2:Euclidean)* specifies areas of triangles from Euclidean space in inches. •

A conceptual *ċon:{-2, -1, 0, 2}* specifies the attribute *ċon(-1)* of the individual *ċon(0)* from the concept space *ċon(2)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, −1:area, 0:ġeoEle, 2:Euclidean)* specifies the area of the geometric element *ġeoEle* from Euclidean space in inches. •

A conceptual *ċon:{−2, −1, 2}* specifies the attribute *ċon(-1)* of the concept space *ċon(2)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, −1:area, 2:Euclidean)* specifies areas of geometric elements from Euclidean space in inches. •

A conceptual *ċon:{-2, 0}* specifies the individual *ċon(0)* in the attribute space *ċon(-2).*

Example. The conceptual *(−2:inch, 0:ġeoEle)* specifies the attributes of the geometric element *ġeoEle* measured in inches. •

A conceptual *ċon:{-2, 0, 1}* specifies the individual *ċon(0)* from the concept *ċon(1)* in the attribute space *ċon(-2)*.

Example. The conceptual *(−2:inch, 0:ġeoEle, 1:triangle)* specifies the attributes of the triangle *ġeoEle* measured in inches. •

A conceptual *ċon:{-2, 1}* specifies the concept *ċon(1)* in the attribute space *ċon(-2)*.

Example. The conceptual *(−2:inch, 1:triangle)* specifies attributes and individuals of triangles measured in inches. •

A conceptual *ċon:{−2, 1, 2}* specifies the concept *ċon(1)* from the concept space *ċon(2)* in the attribute space *ċon(-2)*.

Example. The conceptual *(−2:inch, 1:triangle, 2:Euclidean)* specifies attributes and individuals of triangles from Euclidean space measured in inches. •

A conceptual *ċon:{-2, 2}* specifies the concept space *ċon(2)* in the attribute space *ċon(-2)*.

Example. The conceptual *(−2:inch, 2:Euclidean)* specifies attributes, individuals and kinds of geometric figures from Euclidean space measured in inches. •

A conceptual *ċon:{-2, 0, 2}* specifies the individual *ċon(0)* from the concept space *ċon(2)* in the attribute space *ċon(-2)*.

Example. The conceptual *(−2:inch, 0:ġeoEle, 2:Euclidean)* specifies the geometric element *ġeoEle* and its attributes from Euclidean space measured in inches. •

A conceptual *ċon:{-2, 0, 1, 2}* specifies the individual *ċon(0)* from the concept *ċon(1)* from the concept space *ċon(2)* in the attribute space *ċon(-2)*.

Example. The conceptual *(−2:inch, 0:ġeoEle, 1:triangle, 2:Euclidean)* specifies the triangle *ġeoEle* and its attributes from Euclidean space measured in inches. •

Correlation between other kinds of conceptuals of the order *2* and the corresponding kinds of ontological elements is performed in a similar way.

5.5. Correlation between ontological elements and conceptuals of the order *3* or higher

Correlation between other kinds of conceptuals of the order *3* or higher and the corresponding kinds of ontological elements is performed in a similar way (by the introduction of the attribute space space and so on.).

Example. The conceptual *(−3:10, −2:inch, −1:area, 0:ġeoEle, 1:triangle, 2:Euclidean, 3:2)* specifies the area of the triangle *ġeoEle* from two-dimensional Euclidean space measured in inches in the decimal system. •

6. Modelling of ontological elements

The ontological elements which are directly represented in terms of elements and conceptuals of states were considered in previous sections. The ontological elements which are not directly represented in these terms are modelled in this section.

6.1. Relations and their instances

Binary relations are modelled by direct concepts, and their instances are modelled by elements of the order *0* of these concepts, represented by pairs of elements.

Relations of the arity $\dot{n}at$ are modelled by direct concepts, and their instances are modelled by elements of the order *0* of these concepts, represented by tuples of the length $\dot{n}at$.

Relations of the variable arity are modelled by direct concepts, and their instances are modelled by elements of the order *0* of these concepts, represented by tuples of the variable length.

6.2. Types and domains

Types are modelled by direct concepts, and their values are modelled by elements of the order *0* of these concepts.

Domains as the special kind of types are also modelled by direct concepts, and their values are modelled by elements of the order *0* of these concepts.

Types of attributes of the order $\dot{n}at$ are modelled by the special attribute *type* of the order $\dot{n}at$ + *1*. Values of this attribute are types modelled by direct concepts.

6.3. Inheritance

The usual inheritance relation on concepts is generalized to the inheritance relation on elements of the same order in $[\![\dot{s}ta]\!]$. It is modelled by the special direct concept *inherits*, and their instances are modelled by elements of the order *0* of the concept *inherits*, represented by triples of elements. Elements of the triple specify the inheriting element, the inherited element and their order, respectively.

An element $\dot{e}le[\![\dot{s}ta]\!]$ inherits from $\dot{e}le'[\![\dot{s}ta]\!]$ in $[\![\dot{s}ta, \dot{i}nt]\!]$, if $\dot{s}ta(0{:}(\dot{e}le, \dot{e}le', \dot{i}nt), 1{:}inherits) \neq \omega$.

Inheritance on elements redefines semantics of conceptuals *sem* as follows:

- if $\dot{s}ta(\dot{c}on) \neq \omega$, then $sem(\dot{c}on,\ \dot{s}ta) = \dot{s}ta(\dot{c}on)$;
- if
  - $\dot{s}ta(\dot{c}on) = \omega$,
  - $\dot{i}nt$ is a maximal order in $[\![\dot{c}on]\!]$,
  - $\dot{s}et$ is a set of $\dot{e}le[\![\dot{s}ta]\!]$ such that $\dot{c}on(\dot{i}nt)$ inherits from $\dot{e}le$ in $[\![\dot{s}ta,\ \dot{i}nt]\!]$,
  - $\dot{s}et \neq \emptyset$,
  - $sem(\dot{c}on(\dot{i}nt \leftarrow \dot{e}le),\ \dot{s}ta) = sem(\dot{c}on(\dot{i}nt \leftarrow \dot{e}le'),\ \dot{s}ta)$ for all $\dot{e}le,\ \dot{e}le' \in \dot{s}et$,

  then $sem(\dot{c}on,\ \dot{s}ta) = sem(\dot{c}on(\dot{i}nt \leftarrow \dot{e}le),\ \dot{s}ta)$, where $\dot{e}le \in \dot{s}et$;
- otherwise, $sem(\dot{c}on,\ \dot{s}ta) = \omega$.

The special case of inheritance on direct concepts is defined. A concept $\dot{d}irConc[\![\dot{s}ta]\!]$ inherits from a concept $\dot{d}irConc'[\![\dot{s}ta]\!]$ in $[\![\dot{s}ta]\!]$, if $\dot{d}irConc$ inherits from $\dot{d}irConc'$ in $[\![\dot{s}ta,\ 1]\!]$.

The inheritance relation on elements of the same order in $[\![\dot{s}ta]\!]$ is generalized to the inheritance relation on ordered structures of elements of the same length in $[\![\dot{s}ta]\!]$. The corresponding elements of these structures have the same order. This relation is modelled by the special direct concept *inheritsStr*, and their instances are modelled by elements of the order *0* of this concept, represented by triples of sorted structures of the same length. The elements of the triple specify the ordered structures of inheriting elements, inherited elements and their orders, respectively.

An element $\dot{o}rdStr_1$ inherits from $\dot{o}rdStr_2$ in $[\![\dot{s}ta,\ \dot{o}rdStr]\!]$, if the following properties hold:

- $\dot{o}rdStr = (\dot{i}nt_1,\ ...,\ \dot{i}nt_{\dot{n}at})$;
- $\dot{i}nt_1 < \ldots < \dot{i}nt_{\dot{n}at}$;
- $len(\dot{o}rdStr_1) = len(\dot{o}rdStr_2) = \dot{n}at$;
- $sem((0{:}(\dot{o}rdStr_1,\ \dot{o}rdStr_2,\ \dot{o}rdStr),\ 1{:}inheritsStr),\ \dot{s}ta) \neq \omega$.

Inheritance on ordered structures redefines semantics of conceptuals *sem*:

- if $\dot{s}ta(\dot{c}on) \neq \omega$, then $sem(\dot{c}on,\ \dot{s}ta) = \dot{s}ta(\dot{c}on)$;
- if
  - $\dot{s}ta(\dot{c}on) = \omega$,
  - $\dot{i}nt_1 < \ldots < \dot{i}nt_{\dot{n}at}$ are element orders in $\dot{c}on$,
  - for all $\dot{i}nt$ if $\dot{i}nt \geq \dot{i}nt_1$, and $\dot{i}nt$ is an element order in $\dot{c}on$, then $\dot{i}nt$ coincides with one of the numbers $\dot{i}nt_1,\ ...,\ \dot{i}nt_{\dot{n}at}$,
  - $\dot{s}et$ is a set of $\dot{e}le[\![\dot{s}ta]\!]$ such that $(\dot{c}on(\dot{i}nt_1),\ \ldots,\ \dot{c}on(\dot{i}nt_{\dot{n}at}))$ inherits from $\dot{e}le$ in $[\![\dot{s}ta,\ (\dot{i}nt_1,\ \ldots,\ \dot{i}nt_{\dot{n}at})]\!]$,
  - $\dot{s}et \neq \emptyset$,

– for all $\dot{ele}$, $\dot{ele}' \in \dot{set}$

$$sem(\dot{con}(\dot{int}_1 \leftarrow \dot{ele}(int_1), \ldots, \dot{int}_{\dot{nat}} \leftarrow \dot{ele}(int_{\dot{nat}})), \dot{sta}) =$$
$$sem(\dot{con}(\dot{int}_1 \leftarrow \dot{ele}'(int_1), \ldots, \dot{int}_{\dot{nat}} \leftarrow \dot{ele}'(int_{\dot{nat}})), \dot{sta}),$$

then $sem(\dot{con}, \dot{sta}) = sem(\dot{con}(\dot{int}_1 \leftarrow \dot{ele}(int_1), \ldots, \dot{int}_{\dot{nat}} \leftarrow \dot{ele}(int_{\dot{nat}})), \dot{sta})$, where $\dot{ele} \in \dot{set}$;

- otherwise, $sem(\dot{con}, \dot{sta}) = \omega$.

7. Generic conceptuals

A generic conceptual defines a set of conceptuals satisfying a certain template and sets the default value for these conceptuals. Conceptuals from this set are called instances of the generic conceptual. The template of the generic conceptual is defined by its form.

7.1. The main definitions

Let $* \in ato$. A conceptual $\dot{con}[\![\dot{sta}]\!]$ is called a generic conceptual in $[\![\dot{sta}]\!]$, if there exists $\dot{eleOrd}$ such that $\dot{con}(\dot{eleOrd}) \in \{*, (*, \dot{ele}_2), (*, \dot{ele}_2, \dot{ele}_3), (*, *, \dot{ele}_3)\}$. The element ele of the form $\dot{con}(\dot{eleOrd})$ from this definition is called a substitution place in $[\![\dot{con}, \dot{sta}, \dot{eleOrd}]\!]$. Let $genCon$ and pla be sets of generic conceptuals and substitution places, respectively. The number $\dot{eleOrd}$ is called an order in $[\![\dot{pla}, \dot{con}, \dot{sta}]\!]$. The elements $\dot{ele}_2$ and $\dot{ele}_3$ are called a type and parameter in $[\![\dot{pla}, \dot{con}, \dot{sta}, \dot{eleOrd}]\!]$, respectively. Let $type$ and par be sets of types and parameters in $[\![\dot{pla}, \dot{con}, \dot{sta}, \dot{eleOrd}]\!]$, respectively.

A conceptual $\dot{genCon}$ is called partially typed in $[\![\dot{sta}]\!]$, if there exist $\dot{pla}$, $\dot{type}$ and $\dot{eleOrd}$ such that $\dot{pla}$ is a substitution place in $[\![\dot{genCon}, \dot{sta}, \dot{eleOrd}]\!]$, and $\dot{type}$ is a type in $[\![\dot{pla}, \dot{genCon}, \dot{sta}, \dot{eleOrd}]\!]$.

A conceptual $\dot{genCon}$ is called typed in $[\![\dot{sta}]\!]$, if for all $\dot{pla}$ and $\dot{eleOrd}$, if $\dot{pla}$ is a substitution place in $[\![\dot{genCon}, \dot{sta}, \dot{eleOrd}]\!]$, then there exists $\dot{type}$ such that $\dot{type}$ is a type in $[\![\dot{pla}, \dot{genCon}, \dot{sta}, eleOrd]\!]$.

A conceptual $\dot{genCon}$ is called parametric in $[\![\dot{sta}]\!]$, if there exist $\dot{pla}$, $\dot{par}$ and $\dot{eleOrd}$ such that $\dot{pla}$ is a substitution place in $[\![\dot{genCon}, \dot{sta}, \dot{eleOrd}]\!]$, and $\dot{par}$ is a parameter in $[\![\dot{pla}, \dot{genCon}, \dot{sta}, \dot{eleOrd}]\!]$.

A conceptual $\dot{con}$ is called an instance in $[\![\dot{genCon}, \dot{sta}]\!]$, if the following properties hold:

- if $\dot{genCon}(\dot{int})$ is not a substitution place in $[\![\dot{genCon}, \dot{sta}, \dot{int}]\!]$, then $\dot{con}(\dot{int}) = \dot{genCon}(\dot{int})$;
- if $\dot{genCon}(\dot{int})$ is a substitution place in $[\![\dot{genCon}, \dot{sta}, \dot{int}]\!]$, then $\dot{con}(\dot{int})$ is an element

in ⟦$\dot{s}ta$, $\dot{i}nt$⟧;

- if $\dot{g}enCon(\dot{i}nt) \in \{(*, \dot{t}ype), (*, \dot{t}ype, \dot{p}ar)\}$, then $\dot{c}on(\dot{i}nt)$ is an element in ⟦*conc:*$\dot{t}ype$, $\dot{s}ta$, *concOrd:1*, *eleOrd:0*⟧;
- if $\dot{p}ar$ is a parameter in ⟦$\dot{p}la_1$, $\dot{g}enCon$, $\dot{s}ta$, $\dot{e}leOrd_1$⟧ and ⟦$\dot{p}la_2$, $\dot{g}enCon$, $\dot{s}ta$, $\dot{e}leOrd_2$⟧, then $\dot{c}on(\dot{e}leOrd_1) = \dot{c}on(\dot{e}leOrd_2)$.

A CTS *cts* is called a CTS in ⟦*genCon:*⟧, if the following properties hold:

- **(the consistency property)** if $\dot{g}enCon_1 \neq \dot{g}enCon_2$, then there is no $\dot{c}on$ such that $\dot{c}on$ is an instance of $\dot{g}enCon_1$ and $\dot{g}enCon_2$ in ⟦$\dot{s}ta$⟧;
- semantics of conceptuals *sem* is redefined as follows:
  - if $\dot{s}ta(\dot{c}on) \neq \omega$, then $sem(\dot{c}on, \dot{s}ta) = \dot{s}ta(\dot{c}on)$;
  - if $\dot{s}ta(\dot{c}on) = \omega$, and $\dot{c}on$ is an instance in ⟦$\dot{g}enCon$, $\dot{s}ta$⟧, then $sem(\dot{c}on, \dot{s}ta) = \dot{s}ta(\dot{g}enCon)$;
  - otherwise, $sem(\dot{c}on, \dot{s}ta) = \omega$.

7.2. Examples of generic conceptuals

A conceptual $\dot{g}enCon$*:{-1, 0:*, 1}* specifies the property that the value of the attribute $\dot{g}enCon(-1)$ of individuals from the concept $\dot{g}enCon(1)$ is equal to $\dot{s}ta(\dot{g}enCon)$ in ⟦$\dot{s}ta$⟧, if it is not defined explicitly.

Example. The conceptual $\dot{g}enCon$*:{−1:area, 0:*, 1:triangle}* specifies the property that the area of triangles is equal to $\dot{s}ta(\dot{g}enCon)$ in ⟦$\dot{s}ta$⟧, if it is not defined explicitly. •

A conceptual $\dot{g}enCon$*:{-1, 0:*}* specifies the property that the value of the attribute $\dot{g}enCon(-1)$ of individuals is equal to $\dot{s}ta(\dot{g}enCon)$ in ⟦$\dot{s}ta$⟧, if it is not defined explicitly.

Example. The conceptual $\dot{g}enCon$*:{−1:area, 0:*}* specifies the property that the area of geometric elements is equal to $\dot{s}ta(\dot{g}enCon)$ in ⟦$\dot{s}ta$⟧, if it is not defined explicitly. •

A conceptual $\dot{g}enCon$*:{0:*, 1}* specifies the property that the value of individuals from the concept $\dot{g}enCon(1)$ is equal to $\dot{s}ta(\dot{g}enCon)$ in ⟦$\dot{s}ta$⟧, if it is not defined explicitly.

Example. The conceptual $\dot{g}enCon$*:{0:*, 1:triangle}* specifies the property that the value of triangles is equal to $\dot{s}ta(\dot{g}enCon)$ in ⟦$\dot{s}ta$⟧, if it is not defined explicitly. What is the value of a triangle depends on interpretation. •

7.3. Classification of ontological elements and their properties based on generic conceptuals

Generic conceptuals together with attributes allow to classify ontological elements and their properties in more detail.

A conceptual $\dot{g}enCon$:{-2:type, -1, 0*, 1} specifies the property that the type of the attribute $\dot{g}enCon(-1)$ of individuals from the concept $\dot{g}enCon(1)$ is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for individuals explicitly.

Example. The conceptual $\dot{g}enCon$:{-2:type, -1:area, 0:*, 1:triangle} specifies the property that the type of the attribute *area* of triangles is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for triangles explicitly. •

A conceptual $\dot{g}enCon$:{-2:type, -1, 0:*} specifies the property that the type of the attribute $\dot{g}enCon(-1)$ of individuals is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for individuals explicitly.

Example. The conceptual $\dot{g}enCon$:{-2:type, -1:area, 0:*} specifies the property that the type of the attribute *area* of geometric elements is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for geometric elements explicitly. •

A conceptual $\dot{g}enCon$:{-2:type, 0:*} specifies the property that the type of individuals is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for individuals explicitly.

Example. The conceptual $\dot{g}enCon$:{-2:type, 0:*} specifies the property that the type of geometric elements is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for geometric elements explicitly. •

A conceptual $\dot{g}enCon$:{-2:type, 0:*, 1} specifies the property that the type of individuals from the concept $\dot{g}enCon(1)$ is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for such individuals explicitly.

Example. The conceptual $\dot{g}enCon$:{-2:type, 0:*, 1:triangle} specifies the property that the type of triangles is equal to $\dot{s}ta(\dot{g}enCon)$ in $[\![\dot{s}ta]\!]$, if it is not defined for triangles explicitly. •

8. Justification of requirements for conceptual transition systems

In this section, we establish that CTSs meet the requirements stated in section 1:

1. *The formalism describes the conceptual structure of the specified system.* The conceptual structure of *sys* is described by elements (attributes, concepts, individuals) and, in more detail, usual and generic conceptuals of states of $\dot{c}ts$.
2. *The formalism describes the content of the conceptual structure of the specified system, i. e. it describes the specified system in the context of the conceptual structure.* The content

of the conceptual structure of $\dot{s}ys$ is described by conceptual states of $\dot{c}ts$.

3. *The formalism describes the change of the conceptual structure of the specified system.* The change of the conceptual structure of $\dot{s}ys$ is described by the transition relation *traRel* on conceptual states of $\dot{c}ts$ which specify conceptual structures of $\dot{s}ys$ with different sets of ontological elements.
4. *The formalism describes the change of the content of the conceptual structure of the specified system, i. e. it describes the change of the specified system in the context of the conceptual structure.* The change of the content of the conceptual structure of $\dot{s}ys$ is described by the transition relation *traRel* on conceptual states of $\dot{c}ts$ which specify the same conceptual structure of $\dot{s}ys$. In fact, the distinction between requirements 3 and 4 is relative, for conceptuals allow to define classifications of ontological elements with different granularity.
5. *The formalism is quite universal to specify typical ontological elements (concepts, attributes, concept instances, relations, relation instances, individuals, types, domains, and so on.).* Specification of typical ontological elements is presented in sections 4 and 6.
6. *The formalism provides a quite complete classification of ontological elements, including the determination of their new kinds and subkinds.* Classification of ontological elements based on the two-level scheme is presented in section 5.
7. *The formalism is based on the conception 'state – transition' of the usual transition systems, keeping their simplicity and universality and adding a conceptual 'filling'.* CTSs are the special kind of transition systems in which transitions are defined in the ordinary way, and states are quite simple functions specifying the conceptual structure of the specified systems. Therefore, they keep simplicity and universality of the usual transition systems.
8. *The formalism supports reflection of any order, i. e. allows to specify: the system (reflection of the order 0), the specification of the system (reflection of the order 1), the specification of the specification of the system (reflection of the order 2) and so on. Specifications of the higher order (with reflection of the higher order) impose restrictions on the specifications of the lower order (with reflection of the lower order).* The order of reflection in the specification of $\dot{s}ys$ is defined by the maximal (concretization) order of conceptuals in states of $\dot{c}ts$, i. e. the maximal order of attributes in states of $\dot{c}ts$. The formal description of this property requires additional definitions which are given below.

We extend the (concretization) order of conceptuals on states, the transition relation and CTSs.

A number $\dot{n}at$ is called an order in $[\![\dot{s}ta]\!]$, if the following properties hold:

- there is no $\dot{c}on[\![\dot{s}ta]\!]$ such that the order in $[\![\dot{c}on]\!]$ is greater than $\dot{n}at$;
- there exists $\dot{c}on[\![\dot{s}ta]\!]$ such that $\dot{n}at$ is an order in $[\![\dot{c}on]\!]$.

A state $\dot{s}ta$ is called admissible in $[\![\dot{c}ts]\!]$, if there exists $\dot{s}ta'$ such that *traRel*($\dot{s}ta$, $\dot{s}ta'$), or *traRel*($\dot{s}ta'$, $\dot{s}ta$).

A number $\dot{n}at$ is called an order in $[\![traRel]\!]$, if the following properties hold:

- there is no $\dot{s}ta$ such that $\dot{s}ta$ is admissible in $[\![\dot{c}ts]\!]$, and the order in $[\![\dot{s}ta]\!]$ is greater than $\dot{n}at$;
- there exists $\dot{s}ta$ such that $\dot{s}ta$ is admissible in $[\![\dot{c}ts]\!]$, and $\dot{n}at$ is an order in $[\![\dot{s}ta]\!]$.

A number $\dot{n}at$ is called an order in $[\![\dot{c}ts]\!]$, if $\dot{n}at$ is an order in $[\![traRel[\![\dot{c}ts]\!]]\!]$.

A system $\dot{c}ts$ is called a specification in $[\![\dot{s}ys,\ \dot{n}at0]\!]$, if $\dot{n}at0$ is an order in $[\![\dot{c}ts]\!]$. A number *nat0* is called an order (of specification or reflection) in $[\![\dot{s}ys,\ \dot{c}ts]\!]$.

States, the transition relation and CTSs of greater orders concretize states, the transition relation and CTSs of lower orders, define the special their kinds and are used to classify them.

Thus, the requirements stated in section 1 are met for CTSs.

9. Related formalisms

We compare CTSs with three related formalisms: abstract state machines [1, 2], ontological transition systems [5] and domain-specific transition systems [7]. The comparison is based on the requirements stated in section 1.

9.1. Abstract state machines

Abstract state machines [1, 2] are the special kind of transition systems in which transitions are defined in the ordinary way, and states are algebraic systems. The application of abstract state machine to specifying various systems can be found in [8]. In contrast to CTSs which have no implementation language, abstract state machines have two implementation languages: ASML [9] and XASM [10].

We consider the fulfillment of the requirements for this formalism:

1. *The formalism describes the conceptual structure of the specified system.* The conceptual structure of the specified system is modelled by the appropriate choice of symbols of

the signature of an algebraic system. Thus, both abstract state machines and CTSs describe the conceptual structure of specified systems, but CTSs make it by more natural ontological way.

2. *The formalism describes the content of the conceptual structure of the specified system.* The content of the conceptual structure of the specified system is modelled by the interpretation of signature symbols in a particular state.
3. *The formalism describes the change of the conceptual structure of the specified system.* The change of the conceptual structure of the specified system is described by the transition relation on algebraic structures of different signatures.
4. *The formalism describes the change of the content of the conceptual structure of the specified system.* The change of the content of the conceptual structure of the specified system is described by the transition relation on algebraic structures of the same signature.
5. *The formalism is quite universal to specify typical ontological elements.* In contrast to CTSs, typical ontological elements are not naturally specified by abstract state machines.
6. *The formalism provides a quite complete classification of ontological elements, including the determination of their new kinds and subkinds.* In contrast to CTSs, abstract state machines do not allow to classify naturally ontological elements and define their new kinds and subkinds.
7. *The formalism is based on the conception 'state – transition' of the usual transition systems, keeping their simplicity and universality and adding a conceptual 'filling'.* Abstract state machines are the special kind of transition systems in which transitions are defined in the usual way, and states are functions of algebraic systems. Therefore, they keep simplicity and universality of the usual transition systems sufficiently. The difference between abstract state machines and CTSs consists in that they are based on different (ontological and algebraic, respectively) approaches to the definition of states.
8. *The formalism supports reflection of any order.* In contrast to CTSs, abstract state machines do not support reflection of any order in natural way.

9.2. Ontological transition systems

Ontological transition systems [5] are the special kind of labelled transition systems in which transitions are defined in the usual way, transition labels are actions which change states, and states are ontology content retrieval functions. The ontology in ontological transition systems are defined as a set of concepts and relations on the universe of objects. The content of a concept is defined as a set of sequences of objects from the universe, and the content of a relation is defined as a set of pairs of sequences of objects form the universe. In contrast to CTSs, ontological transition systems have their associated notation language OTSL [5, 6] which specifies states and transition actions.

We consider the fulfillment of the requirements for this formalism:

1. *The formalism describes the conceptual structure of the specified system.* Ontological transition systems describe the conceptual structure of the specified system by concepts and relations.
2. *The formalism describes the content of the conceptual structure of the specified system.* Ontological transition systems describe the content of the conceptual structure of the specified system by ontology content retrieval functions.
3. *The formalism describes the change of the conceptual structure of the specified system.* In contrast to CTSs, ontological transition sytems do not change the conceptual structure of the specified system.
4. *The formalism describes the change of the content of the conceptual structure of the specified system.* The change of the content of the conceptual structure of the specified system is described by actions of the corresponding ontological transition system.
5. *The formalism is quite universal to specify typical ontological elements.* In contrast to CTSs, only some typical ontological elements are naturally specified by ontological transition systems.
6. *The formalism provides a quite complete classification of ontological elements, including the determination of their new kinds and subkinds.* In contrast to CTSs, a set of kinds of ontological elements which are naturally defined by ontological transition systems is restricted.
7. *The formalism is based on the conception 'state – transition' of the usual transition systems, keeping their simplicity and universality and adding a conceptual 'filling'.* Ontological transition systems are the special kind of transition systems in which transitions are defined in the usual way, and states are quite simple ontology content retrieval functions. Therefore, they keep simplicity and universality of the usual transition systems sufficiently. The language of specification of ontological transition systems OTSL includes a specific set of actions that restrict the transition relation. Therefore, it is not as much

universal as CTSs.

8. *The formalism supports reflection of any order.* In contrast to CTSs, ontological transition systems do not support reflection.

9.3. Domain-specific transition systems

Domain-specific transition systems [7] are the special kind of transition systems in which states are defined by parametric forms and their values, and transitions are defined by the special kind of these forms – transition rules. A parametric form is characterized by a sample, evaluated and quoted parameters, a parameter constraint, a return value constraint and rules of propagation of indeterminate values of parameters. Instances of the form are defined by the pattern matching algorithm. Applying transition rules are described by the algorithms of pattern matching and rule execution.

We consider the fulfillment of the requirements for this formalism:

1. *The formalism describes the conceptual structure of the specified system.* The conceptual structure of the specified system is described by the special kind of domain-specific transition systems – ontological domain-specific transition systems. Concepts in such systems are modelled by forms which represent characteristic functions of these concepts and define their content. The sample of a rule of such system is represented by a parameter, and the parameter constraint defines the concept for which values of the parameter are instances.
2. *The formalism describes the content of the conceptual structure of the specified system.* Ontological domain-specific transition systems describe the content of the conceptual structure of the specified system by concepts.
3. *The formalism describes the change of the conceptual structure of the specified system.* In contrast to CTSs, domain-specific transition systems do not change the conceptual structure of the specified system.
4. *The formalism describes the change of the content of the conceptual structure of the specified system.* The change of the content of the conceptual structure of the specified system is described by rules of the corresponding domain-specific transition system.
5. *The formalism is quite universal to specify typical ontological elements.* In contrast to CTSs, only concepts and their instances are naturally specified by ontological domain-specific transition systems.

6. *The formalism provides a quite complete classification of ontological elements, including the determination of their new kinds and subkinds.* In contrast to CTSs, domain-specific transition systems do not allow to classify ontological elements and define their new kinds.
7. *The formalism is based on the conception 'state – transition' of the usual transition systems, keeping their simplicity and universality and adding a conceptual 'filling'.* Domain-specific transition systems are the special kind of transition systems in which transitions are defined by sets of transition rules, the quite general pattern matching algorithm which specifies applicability of these rules, and the specific rule execution algorithm, and states are functions on forms with specific attributes. Therefore, domain-specific transition systems do not keep simplicity and universality of the usual transition systems sufficiently.
8. *The formalism supports reflection of any order.* In contrast to CTSs, domain-specific transition systems do not support reflection.

10. Conclusion

In the paper, the main definitions of the theory of CTSs were given, classifications for elements of states of CTSs and for conceptuals and their associated ontological elements were developed, generalization of conceptuals which allows to make more comprehensive classification of ontological elements – generic conceptuals – was proposed.

We plan to develop the special kinds of CTSs concretizing the transition relation, and the language of specification of CTSs that would describe the transition relation concretizations which are important for practical purposes.

Development of formal methods based on this language to solve problems of designing and prototyping software systems as well as specification of operational and axiomatic semantics of programming languages is an important application of CTSs. In the case of specification of operational semantics of a programming language, a CTS specifies the abstract machine of the language. In the case of specification of axiomatic semantics of a programming language, a CTS specifies a generator of verification conditions for programs in the language, based on its axiomatic semantics.

We hope that the use of the ontological approach will reduce the gap between the great potential of formal methods and, with rare exceptions, toy examples of their application for solving the above problems [11].

Because of their simplicity and universality, CTSs can be also used for classification and formalization of concepts, properties and algorithms in the field of ontology evolution [12–15], as well as designing and prototyping tools to support ontology evolution [16].

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